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BRADBURY'S

ELEMENTARY ALGEBRA,

DESIGNED FOR THE

USE OF HIGH SCHOOLS AND ACADEMIES.

BY

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PREFACE.

It was the intention of the author of Eaton's Arithmetics to add to the series an Algebra, and he had commenced the preparation of such a work. Although its completion has devolved upon another, the author, as far as practicable in a work of this character, has followed the same general plan that has made the Arithmetics so popular, and spared no labor to adapt the book to the wants of pupils commencing this branch of mathematics.

A few problems have been introduced in Section II., to awaken the pupil's interest in Algebraic operations, and thus prepare him for the more abstract principles which must be mastered before the more difficult problems can be solved. Special attention is invited to the arrangement of the equations in Elimination; to the Second Method of Completing the Square in Affected Quadratics; and to the number and variety of the examples given in the body of the work and in the closing section.

The Theory of Equations, the Explanation of Negative Results, of Zero and Infinity, and of Imaginary Quantities, are omitted, as topics not appropriate to an Elementary Algebra. It may also be better for the younger pupils to

pass over the two theorems in Art. 74, until they become more familiar with algebraic reasoning.

While the book has not been made simple by avoiding the legitimate use of the negative sign before a parenthesis or a fraction, the difficulty which is caused to beginners by the introduction of negative indices in simple division has been obviated by deferring their introduction to the section on Powers and Roots, where they are fully explained.

The utmost conciseness consistent with perspicuity has been studied throughout the work. It is hoped the book will commend itself to both teachers and pupils.

W. F. B.

CAMBRIDGE, MASS., May 17, 1868.

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ELEMENTARY ALGEBRA.

SECTION I.

DEFINITIONS.

1. Mathematics is the science of quantity.

2. Quantity is that which can be measured; as distance, time, weight.

3. Arithmetic is the science of numbers. In Arithmetic quantities are represented by figures.

4. Algebra is Universal Arithmetic. In Algebra quantities are represented by either letters or figures, and their relations by signs.

NOTATION.

5. Addition is denoted by the sign $+$, called *plus;* thus, $3 + 2$, i. e. 3 plus 2, signifies that 2 is to be added to 3.

6. Subtraction is denoted by the sign $-$, called *minus;* thus, $7 - 4$, i. e. 7 minus 4, signifies that 4 is to be subtracted from 7.

7. Multiplication is denoted by the sign $\times$; thus, 6×5 signifies that 6 and 5 are to be multiplied together. Between a figure and a letter, or between letters, the sign $\times$ is generally omitted; thus, $6ab$ is the same as $6 \times a \times b$. Multiplication is sometimes denoted by the period; thus, $8 . 6 . 4$ is the same as $8 \times 6 \times 4$.

8. Division is denoted by the sign $\div$; thus, $9 \div 3$ signifies that 9 is to be divided by 3. Division is also indicated by the fractional form; thus, $\frac{9}{3}$ is the same as $9 \div 3$.

9. Equality is denoted by the sign $=$; thus, $\$1 = 100$ cents, signifies that 1 dollar is equal to 100 cents. An expression in which the sign $=$ occurs is called an equation, and that portion which precedes the sign $=$ is called the *first member*, and that which follows, the *second member*.

10. Inequality is denoted by the sign $>$ or $<$, the smaller quantity always standing at the vertex; thus, $8 > 6$ or $6 < 8$ signifies that 8 is greater than 6.

11. Three dots $\therefore$ are sometimes used, meaning *hence, therefore.*

12. A Parenthesis ( ), or a Vinculum ———, indicates that all the quantities included, or connected, are to be considered as a single quantity, or to be subjected to the same operation; thus, $(8 + 4) \times 3 = 12 \times 3$, or $= 24 + 12 = 36$; $\overline{21 - 6} \div 3 = 15 \div 3$, or $= 7 - 2 = 5$. Without the parenthesis, these examples would stand thus: $8 + 4 \times 3 = 8 + 12 = 20$; $21 - 6 \div 3 = 21 - 2 = 19$; the sign $\times$, in the former, not affecting 8; nor the sign $\div$, in the latter, 21.

Examples.

1. $9 + 7 - 3 + 4 =$ how many?
2. $(9 + 15) \div 3 =$ how many?
3. $\dfrac{17 + 13}{10} \times 14 =$ how many?
4. $(14 + 13) \times (5 - 2) =$ how many?
5. $10 + \overline{(7 - 4) \div 3} \times 4 =$ how many?
6. $25 - (6 + 7) =$ how many?
7. $150 - (18 - 11) =$ how many?

8. Prove that $175 + 8 - 49 = 14 + 190 - 54 - 16$.

9. Prove that $216 - 44 + 14 > 144 + 13 - 75$.

10. Place the proper sign ($=$, $>$, or $<$) between these two expressions, $(247 + 104)$ and $(546 - 195)$.

11. Place the proper sign ($=$, $>$, or $<$) between these two expressions, $(119 - 47 + 16)$ and $(317 - 104)$.

12. Place the proper sign ($=$, $>$, or $<$) between these two expressions, $(417 + 31) - (187 - 72)$ and $(127 + 179)$.

AXIOMS.

13. All operations in Algebra are based upon certain self-evident truths called Axioms, of which the following are the most common: —

1. If equals are added to equals the sums are equal.
2. If equals are subtracted from equals the remainders are equal.
3. If equals are multiplied by equals the products are equal.
4. If equals are divided by equals the quotients are equal.
5. Like powers and like roots of equals are equal.
6. The whole of a quantity is greater than any of its parts.
7. The whole of a quantity is equal to the sum of all its parts.
8. Quantities respectively equal to the same quantity are equal to each other.

SECTION II.

ALGEBRAIC OPERATIONS.

14. A Theorem is something to be proved.

15. A Problem is something to be done.

16. The Solution of a Problem in Algebra consists,—

1st. In reducing the statement to the form of an equation;

2d. In reducing the equation so as to find the value of the unknown quantities.

Examples for Practice.

1. The sum of the ages of a father and his son is 60 years, and the age of the father is double that of the son; what is the age of each?

It is evident that if we knew the age of the son, by doubling it we should know the age of the father. Suppose we let x equal the age of the son; then $2x$ equals the age of the father; and then, by the conditions of the problem, x, the son's age, plus $2x$, the father's age, equals 60 years; or $3x$ equals 60, and (Axiom 4) x, the son's age, is $\frac{1}{3}$ of 60, or 20, and $2x$, the father's age, is 40. Expressed algebraically, the process is as follows:—

$$\begin{aligned} \text{Let} \quad x &= \text{son's age,} \\ \text{then} \quad 2x &= \text{father's age.} \\ \hline x + 2x &= 60, \\ 3x &= 60, \\ x &= 20, \text{ the son's age.} \\ 2x &= 40, \text{ the father's age.} \end{aligned}$$

2. A horse and carriage are together worth \$450; but the horse is worth twice as much as the carriage; what is each worth? Ans. Carriage, \$150; horse, \$300.

All problems should be verified to see if the answers obtained fulfil the given conditions. In each of the preceding problems there are two conditions, or statements. For example, in Prob. 2 it is stated (1st) that the horse and carriage are together worth \$450, and (2d) that the horse is worth twice as much as the carriage: both these statements are fulfilled by the numbers 150 and 300.

3. The sum of two numbers is 72, and the greater is seven times the less; what are the numbers?

4. A drover being asked how many sheep he had, said that if he had ten times as many more, he should have 440; how many had he?

5. A father and son have property of the value of \$8015, and the father's share is four times the son's; what is the share of each?

Ans. Father's, \$6412; son's, \$1603.

6. A farmer has a horse, a cow, and a sheep; the horse is worth twice as much as the cow, and the cow twice as much as the sheep, and all together are worth \$490; how much is each worth?

OPERATION.

Let $x =$ the price of the sheep,

then $2x =$ " " " " cow,

and $4x =$ " " " " horse;

and their sum $7x = 490$,

$x = 70$, the price of the sheep,

and $2x = 140$, " " " " cow,

and $4x = 280$, " " " " horse.

7. A man has three horses which are together worth \$540, and their values are as the numbers 1, 2, and 3; what are the respective values?

Let x, $2x$, and $3x$ represent the respective values.

Ans. \$90, \$180, and \$270.

8. A man has three pastures, containing 360 sheep, and the numbers in each are as the numbers 1, 3, and 5; how many are there in each?

9. Divide 63 into three parts, in the proportion of 2, 3, and 4.

Let $2x$, $3x$, and $4x$ represent the parts.

10. A man sold an equal number of oxen, cows, and sheep for \$1500; for an ox he received twice as much as for a cow, and for a cow eight times as much as for a sheep, and for each sheep \$6; how many of each did he sell, and what did he receive for all the oxen?

Ans. 10 of each, and for the oxen, \$960.

11. Three orchards bore 872 bushels of apples; the first bore three times as many as the second, and the third bore as many as the other two; how many bushels did each bear?

12. A boy spent \$4 in oranges, pears, and apples; he bought twice as many pears and five times as many apples as oranges; he paid 4 cents for each pear, 3 for each orange, and 1 for each apple; how many of each did he buy, and how much did he spend for oranges? how much for pears, and how much for apples?

Ans. { 25 oranges, 50 pears, and 125 apples.
Spent for oranges, \$0.75; pears, \$2; apples, \$1.25.

13. A farmer hired a man and two boys to do a piece of work; to the man he paid \$12, to one boy \$6, and to the other \$4 per week; they all worked the same time, and received \$264; how many weeks did they work?

Ans. 12 weeks.

14. Three men, A, B, and C, agreed to build a piece of wall for \$99; A could build 7 rods, and B 6, while C could build 5; how much should each receive?

15. Four boys, A, B, C, and D, in counting their money, found they had together \$1.98, and that B had twice as much as A, C as much as A and B, and D as much as B and C; how much had each?

Ans. A 18 cents, B 36, C 54, and D 90.

16. It is required to divide a quantity, represented by a, into two parts, one of which is double the other.

OPERATION.

Let $x =$ one part,

then $2x =$ the other part.

$$3x = a,$$

$$x = \frac{a}{3}, \text{ one part,}$$

$$2x = \frac{2a}{3}, \text{ the other part.}$$

17. If in the preceding example $a = 24$, what are the required parts?

Ans. $\frac{a}{3} = \frac{24}{3} = 8$, and $\frac{2a}{3} = \frac{48}{3} = 16$.

18. It is required to divide c into three parts so that the first shall be one half of the second and one fifth of the third.

Ans. $\frac{c}{8}$, $\frac{2c}{8}$, and $\frac{5c}{8}$.

19. Divide n into three parts, so that the first part shall be one third the second and one seventh of the third.

20. A is one half as old as B, and B is one third as old as C, and the sum of their ages is p; what is the age of each?

Ans. A's $\frac{p}{9}$, B's $\frac{2p}{9}$, and C's $\frac{6p}{9}$.

SECTION III.

DEFINITIONS AND NOTATION.

[Continued from Section I.]

17. The last letters of the alphabet, x, y, z, &c., are used in algebraic processes to represent *unknown* quantities, and the first letters, a, b, c, &c., are often used to represent *known* quantities.

Numerical Quantities are those expressed by figures, as 4, 6, 9.

Literal Quantities are those expressed by letters, as a, x, y.

Mixed Quantities are those expressed by both figures and letters, as $3a$, $4x$.

18. The sign plus, $+$, is called the *positive* or *affirmative* sign, and the quantity before which it stands a *positive* or *affirmative* quantity. If no sign stands before a quantity, $+$ is always understood.

19. The sign minus, $-$, is called the *negative* sign, and the quantity before which it stands, a *negative* quantity.

20. Sometimes both $+$ and $-$ are prefixed to a quantity, and the sign and quantity are both said to be *ambiguous*; thus, $8 \pm 3 = 11$ or 5, and $a \pm b = a + b$, or $a - b$, according to circumstances.

21. The words *plus* and *minus*, *positive* and *negative*, and the signs $+$ and $-$, have a merely *relative* signification; thus, the navigator and the surveyor always represent their northward and eastward progress by the sign $+$, and their southward and westward progress by the sign $-$, though, in the nature of things, there is nothing to prevent representing northings and eastings by $-$, and southings and westings by $+$. So if a man's *prop-*

erty is considered *positive*, his *gains* should also be considered *positive*, while his *debts* and his *losses* should be considered *negative*; thus, suppose that I have a farm worth \$5000 and other property worth \$3000 and that I owe \$1000, then the net value of my estate is \$5000 + \$3000 — \$1000 = \$7000. Again, suppose my farm is worth \$5000 and my other property \$3000, while I owe \$12000, then my net estate is worth \$5000 + \$3000 — \$12000 = — \$4000, i. e. I am worth — \$4000, or, in other words, I owe \$4000 more than I can pay. From this last illustration we see that the sign — may be placed before a quantity standing alone, and it then merely signifies that the quantity is negative, without determining what it is to be subtracted from.

22. The TERMS of an algebraic expression are the quantities which are separated from each other by the signs $+$ or $-$; thus, in the equation $4a - b = 3x + c - 7y$, the first member consists of the two terms $4a$ and $-b$, and the second of the three terms $3x$, c, and $-7y$.

23. A COEFFICIENT is a number or letter prefixed to a quantity to show how many times that quantity is to be taken; thus, in the expression $4x$, which equals $x + x + x + x$, the 4 is the coefficient of x; so in $3ab$, which equals $ab + ab + ab$, 3 is the coefficient of ab; in $4ab$, $4a$ may be considered the coefficient of b, or $4b$ the coefficient of a, or a the coefficient of $4b$.

Coefficients may be *numerical* or *literal* or *mixed*; thus, in $4ab$, 4 is the numerical coefficient of ab, a is the literal coefficient of $4b$, $4a$ is the mixed coefficient of b.

If no numerical coefficient is expressed, a *unit* is understood; thus, x is the same as $1x$, bc as $1bc$.

24. An INDEX or EXPONENT is a number or letter placed after and a little above a quantity to show how many times that quantity is to be taken *as a factor;* thus, in the ex-

pression b^3, which equals $b \times b \times b$, the 3 is the index or exponent of the power to which b is to be raised, and it indicates that b is to be used as a factor 3 times.

An exponent, like a coefficient, may be numerical, literal, or mixed; thus, x^3, x^n, x^{5n}, &c.

If no exponent is written, a *unit* is understood; thus $b = b^1$, $a = a^1$, &c.

Coefficients and *Exponents* must be carefully distinguished from each other. A *Coefficient* shows the number of times a quantity is taken to make up a given *sum;* an *Exponent* shows how many times a quantity is taken *as a factor* to make up a given *product;* thus $4x = x + x + x + x$, and $x^4 = x \times x \times x \times x$.

25. The product obtained by taking a quantity as a factor a given number of times is called a *power*, and the exponent shows the *number* of times the quantity is taken.

26. A ROOT of any quantity is a quantity which, taken as a factor a given number of times, will produce the given quantity.

A ROOT is indicated by the radical sign, $\sqrt{}$, or by a fractional exponent. When the radical sign, $\sqrt{}$, is used, the index of the root is written at the top of the sign, though the index denoting the second or square root is generally omitted; thus,

$\sqrt{x}$, or $x^{\frac{1}{2}}$, means the second root of x;
$\sqrt[3]{x}$, or $x^{\frac{1}{3}}$, " " third " " x, &c.

Every quantity is considered to be both the first power and the first root of itself.

27. The RECIPROCAL of a quantity is a unit divided by that quantity. Thus, the reciprocal of 5 is $\frac{1}{5}$, and of x, $\frac{1}{x}$.

28. A Monomial is a single term; as a, or $3x$, or $5bxy$.

29. A Polynomial is a number of terms connected with each other by the signs plus or minus; as $x + y$, or $3a + 4x - 7aby$.

30. A Binomial is a polynomial of two terms; as $3x + 3y$, or $x - y$.

31. A Residual is a binomial in which the two terms are connected by the minus sign, as $x - y$.

32. Similar Terms are those which have *the same powers of the same letters*, as x and $3x$, or $5ax^3$ and $-2ax^3$. But x and x^2, or $5a$ and $5b$, are *dissimilar*.

33. The Degree of a term is denoted by the sum of the exponents of all the literal factors. Thus, $2a$ is of the first degree; $3a^2$ and $4ab$ are of the second degree; and $6a^3x^4$ is of the seventh degree.

34. Homogeneous Terms are those of the same degree. Thus, $4a^2x$, $3abc$, x^2y, are homogeneous with each other.

35. To find the numerical value of an algebraic expression when the literal quantities are known, we must substitute the given values for the letters, and perform the operations indicated by the signs.

The numerical value of $7a - b^4 + c^2$ when $a = 4$, $b = 2$, and $c = 5$ is $7 \times 4 - 2^4 + 5^2 = 28 - 16 + 25 = 37$.

Examples.

Find the numerical values of the following expressions, when $a = 2$, $b = 13$, $c = 4$, $d = 15$, $m = 5$, and $n = 7$.

1. $a + b - c + 2d$. Ans. 41.
2. $a^2 + 3bc - 2cd$. Ans. 40.
3. $\dfrac{4a^3 + m^4}{n - c}$. Ans. 219.

4. $m^2 - 2mn + n^2$.

5. $\frac{c^3}{8} - (d^2 - n^2)$.

6. $(a^2 - c + b)(m + n)$.

7. $\frac{b^3 - c^3}{n - c} \times (d - m + n)$. Ans. 867.

8. $\sqrt{c} + \sqrt[3]{4d + c} - \sqrt{5m}$. Ans. 1.

9. $3a\sqrt{b - c} \times 4n\sqrt[3]{25m}$.

10. $\frac{5m - 6n + 3d}{d - 2c}$. Ans. 4.

11. $\sqrt[3]{d - n} + \sqrt{7n}$.

12. $(b - a)(d - c) - m$. Ans. 116.

13. $13(4d) + 4d - 7a$.

14. $4ab + \sqrt{100c} - \sqrt[3]{d - n}$. Ans. 122.

15. $4a\sqrt{60d} + 5a^2b^2$.

16. $b - a - (d - n)$. Ans. 3.

17. $b - a - d - n$. Ans. -11.

18. $(b - a)(d - n)$. Ans. 88.

19. $(b - a)d - n$. Ans. 158.

20. $a + b\sqrt{10(d - m)} + 14\sqrt{c}$.

36. Write in algebraic form: —

1. The sum of a and b minus the difference of m and n. $(m > n.)$

2. Four times the square root of the sum of a, b, and c.

3. Six times the product of the sum and difference of c and d. $(c > d.)$

4. Five times the cube root of the sum of a, m, and n.

5. The sum of m and n divided by their difference.

6. The fourth power of the difference between a and m.

SECTION IV.

ADDITION.

37. Addition in Algebra is the process of finding the aggregate or sum of several quantities.

For convenience, the subject is presented under three cases.

CASE I.

38. When the terms are similar and have like signs.

1. Charles has 6 apples, James 4 apples, and William 5 apples; how many apples have they all?

OPERATION.

6 apples, 4 apples, 5 apples, 15 apples,	or, letting a represent one apple,	$6\ a$ $4\ a$ $5\ a$ $15\ a$

It is evident that just as 6 apples and 4 apples and 5 apples added together make 15 apples, so $6\ a$ and $4\ a$ and $5\ a$ added together make $15\ a$.

In the same way $-6a$ and $-4a$ and $-5a$ are equal together to $-15a$.

Therefore, when the terms are similar and have like signs:

RULE.

Add the coefficients, and to their sum annex the common letter or letters, and prefix the common sign.

(2.)	(3.)	(4.)	(5.)	(6.)	(7.)
$5\,a\,x$	$3\,a^2$	$4\,x$	$6\,y$	$-3\,x^3$	$-5\,b\,y$
$8\,a\,x$	$4\,a^2$	x	$10\,y$	$-2\,x^3$	$-2\,b\,y$
$4\,a\,x$	$7\,a^2$	$5\,x$	y	$-7\,x^3$	$-b\,y$
$2\,a\,x$	$3\,a^2$	$3\,x$	$2\,y$	$-4\,x^3$	$-b\,y$
$19\,a\,x$		$13\,x$		$-16\,x^3$	

8. What is the sum of $a x^2$, $3 a x^2$, $2 a x^2$, and $4 a x^2$?
Ans. $10 a x^2$.

9. What is the sum of $3 b x$, $4 b x$, $6 b x$, and $b x$?

10. What is the sum of $2 x y$, $6 x y$, $10 x y$, and $8 x y$?

11. What is the sum of $-7 x z$, $-x z$, $-4 x z$, and $-x z$?
Ans. $-13 x z$.

12. What is the sum of $-2 b$, $-3 b$, $-6 b$, and $-3 b$?

13. What is the sum of $-a b c$, $-3 a b c$, $-4 a b c$, and $-a b c$?

CASE II.

39. When the terms are similar and have unlike signs.

1. A man earns 7 dollars one week, and the next week earns nothing and spends 4 dollars, and the next week earns 6 dollars, and the fourth week earns nothing and spends 3 dollars; how much money has he left at the end of the fourth week?

If what he earns is indicated by $+$, then what he spends will be indicated by $-$, and the example will appear as follows:—

OPERATION.

$$\left.\begin{array}{l} +7 \text{ dollars,} \\ -4 \text{ dollars,} \\ +6 \text{ dollars,} \\ -3 \text{ dollars,} \\ \hline +6 \text{ dollars,} \end{array}\right\} \begin{array}{c} \text{or, letting } d \\ \text{represent} \\ \text{one dollar,} \end{array} \left\{\begin{array}{l} +7\,d \\ -4\,d \\ +6\,d \\ -3\,d \\ \hline +6\,d \end{array}\right.$$

Earning 7 dollars and then spending 4 dollars, the man would have 3 dollars left; then earning 6 dollars, he would have 9 dollars; then spending 3 dollars, he would have left 6 dollars; or he earns in all 7 dollars $+$ 6 dollars $=$ 13 dollars; and spends 4 dollars $+$ 3 dollars $=$ 7 dollars; and therefore has left the difference between 13 dollars and 7 dollars $=$ 6 dollars; hence the sum of $+7 d$, $-4 d$, $+6 d$, and $-3 d$ is $+6 d$.

Therefore, when the terms are similar, and have unlike signs:

RULE.

Find the difference between the sum of the coefficients of the positive terms, and the sum of the coefficients of the negative terms, and to this difference annex the common letter or letters, and prefix the sign of the greater sum.

(2.)	(3.)	(4.)	(5.)
$3xy$	$4y^2$	$13abc^2$	$14x^2y$
xy	$-2y^2$	$6abc^2$	$-13x^2y$
$-5xy$	$7y^2$	$-abc^2$	$10x^2y$
$7xy$	$-3y^2$	$-8abc^2$	$-x^2y$
$-2xy$	$14y^2$	$4abc^2$	$24x^2y$
$4xy$		$14abc^2$	

(6.)	(7.)
$25xyz$	$8(x+y)$
$-50xyz$	$-4(x+y)$
$10xyz$	$7(x+y)$
$-67xyz$	$-3(x+y)$
$8xyz$	$-(x+y)$
$-74xyz$	$7(x+y)$

8. Find the sum of $8x^2y^2$, $-14x^2y^2$, $17x^2y^2$, and $-x^2y^2$.

9. Find the sum of $7(x+y)$, $8(x+y)$, $-(x+y)$, and $4(x+y)$. Ans. $18(x+y)$.

10. Find the sum of $-ax^2$, $+ax^2$, $-10ax^2$, $+25ax^2$, and $-13ax^2$.

11. Find the sum of $27ab$, $-34ab$, $-150ab$, $27ab$, and $-13ab$. Ans. $-143ab$.

12. Find the sum of ax^3, $-14ax^3$, $17ax^3$, $-ax^3$, $44ax^3$, and $-ax^3$.

13. Find the sum of $17(a+b)$, $-(a+b)$, $(a+b)$, and $-13(a+b)$. Ans. $4(a+b)$.

CASE III.

40. To find the sum of *any* algebraic quantities.

The sum of $5a$ and $6b$ is neither $11a$, nor $11b$, and can only be expressed in the form of $5a+6b$, or $6b+5a$; and the sum of $5a$ and $-4b$ is $5a-4b$; but in finding the sum of $5a$, $6b$, $5a$, and $-4b$, the a's can be added together by Case I., and the b's by Case II., and the two results connected by the proper sign; thus, $5a+6b+5a-4b=10a+2b$.

1. Find the sum of $6d$, $-2b$, x, $3y$, $5x$, $-3b$, $3bc+4d$, $5x$, $7b+2x$, and $-3bc$.

OPERATION.

$$\begin{array}{rrrrr}
6d & -2b & +x & +3y & +3bc \\
4d & -3b & +5x & & -3bc \\
 & +7b & +5x & & \\
 & & +2x & & \\
\hline
10d & +2b & +13x & +3y &
\end{array}$$

For convenience, similar terms are written under each other; then by Case I. the first column at the left is added; the second by Case II., and so on; $+3bc$ and $-3bc$ cancel.

This case includes the two preceding cases, and hence to find the sum of *any* algebraic quantities:

RULE.

Write similar terms under each other, find the sum of each column, and connect the several sums with their proper signs.

(2.)

$$\begin{array}{rrrrrr}
4x & -7a & +3y & -4b & +3z & \\
 & 6a & -y & +4b & -2z & \\
 & 4a & -2y & +8b & -z & \\
 & -3a & & -8b & & -10c \\
\hline
4x & & & & & -10c
\end{array}$$

(3.)

$$\begin{array}{r}
7a + 4b - \ \ 3c + 2\sqrt{x} \\
-3b + \ \ 3c - 7\sqrt{x} + y \\
-10c + 8\sqrt{x} - y \\
3\sqrt{x} \\
\hline
7a + \ \ b - 10c + 6\sqrt{x}
\end{array}$$

4. Add together $7\sqrt{x}$, $-8x$, $7x^2$, $-6\sqrt{x}$, $4x^2$, $-8x$, $4x$, and $7x^2$, $-8\sqrt{x}$.

Ans. $18x^2 - 12x - 7\sqrt{x}$.

5. Add together $3ax - 4ab + 2xy$, $7ab + 5x - 4a$, $7xy - 3a + 4x$, and $+abc - ax + 6xy$.

6. Add together $7x - 3ay - 5ab + 4c$, $3ax + 4x + 5ab - 5c$, and $3c - 3ax + 7y + c$.

Ans. $11x - 3ay + 3c + 7y$.

7. Add together $5a - 32 + 7x + 4ax - 3ab$, $5ab - 5a + 22 - 4ax + 4$, and $6 - 2ab + 3x + 4y + 4ax$.

8. Add together $6xy + 6xz - 6mn + 4n$, $4mn - 3xy + 2n - 8mn$, $-6xz + 4n - 3xy + 6$, and $10mn - 10n + 3 - 9$.

Ans. 0.

9. Add together $8am + 19nx - 55b + c$, $-19v + 14b - 16c + y$, and $18nx - 44am + 15v - 4y$.

10. Add together $17ax^2 + 19ax^3 - 14ax^4 + 16ax^2$, $13ax^3 - 5ax^4 + 6ax^2 - 10ax^3$, and $14ax^4 + 17ax^2 - 3ax^3 + 15ax^2$.

Ans. $71ax^2 + 19ax^3 - 5ax^4$.

11. Add together $m + n - 4a + 6c - 7y$, $8c - 4m + 3n - 5a + 3c$, $7a - 17c + 7y - 10m - 6n$, and $14n - 8a - 7c + 10y - 8m$.

12. Add together $8axy + 17bxy - 16cxy - 9axy$, $16bxy - 18cxy + 10axy - 14axz$, $16cxy + 25axy - 7bxy + 25cxy$, and $10axz + 3bxy - 10cxy + 4axz$.

Ans. $34axy + 29bxy - 3cxy$.

13. Add together $3(x+y)$, $-4(x+y)$, and $7(x+y)$.
Ans. $6(x+y)$.

14. Add together $5(2x+y-3z)$, and $-2(2x+y-3z)$.

NOTE.—If several terms have a common letter or letters, the sum of their coefficients may be placed in parenthesis, and the common letter or letters annexed; thus,

$$6x+8x-5x=(6+8-5)x;$$
$$ax+3bx-2cx=(a+3b-2c)x;$$
$$bcxy+adxy-acxy=(bc+ad-ac)xy.$$

15. Add together $ax-bx+3x$, and $2ax+4bx-x$.
Ans. $(3a+3b+2)x$.

16. Add together $by-3cy+5ay$, and $cy+4by-2ay$.

17. Add together $2xy-axy$, and $6xy-3axy$.
Ans. $(8-4a)xy$.

18. Add together $7(3x+5y)+3a-6x+8ab$, $3x+5(3x+5y)+7a-5ab$, and $8x+2(3x+5y)-7a-3ab$.
Ans. $47x+70y+3a$.

41. From what has gone before, it will be seen that addition in Algebra differs from addition in Arithmetic. In Arithmetic the quantities to be added are always considered positive; while in Algebra both positive and negative quantities are introduced. In Arithmetic addition always implies augmentation; while in Algebra the sum may be numerically less than any of the quantities added; thus, the sum of $10x$ and $-8x$ is $2x$, which is the *numerical* difference of the two quantities.

SECTION V.

SUBTRACTION.

42. SUBTRACTION in Algebra is finding the difference between two quantities.

1. John has 6 apples and James has 2 apples; how many more has John than James?

Let a represent one apple, and we have

$$\left.\begin{array}{r} 6\,a \\ 2\,a \\ \hline 4\,a \end{array}\right\}, \text{ or } 6\,a - 2\,a = 4\,a.$$

2. During a certain day A made 9 dollars and B lost 6 dollars; what was the difference in the profits of A and B for the day? If gain is considered +, then loss must be considered —, and letting d represent one dollar, it is required to take $-6\,d$ from $9\,d$.

OPERATION.

$$\begin{array}{r} 9\,d \\ -6\,d \\ \hline 15\,d \end{array}$$

It is evident that the difference between A's and B's profits for the day is $9\,d + 6\,d = 15\,d$; that is, $9\,d - (-6\,d) = 9\,d + 6\,d = 15\,d$.

Hence it appears that, as addition does not always imply augmentation, so subtraction does not always imply diminution.

Subtracting a positive quantity is equivalent to adding an equal negative quantity; and subtracting a negative quantity is equivalent to adding an equal positive quantity.

Suppose I am worth \$1000; it matters not whether a thief steals \$400 from me, or a rogue having the authority involves me in debt \$400 for a worthless article; for

in either case I shall be worth only \$600. The thief *subtracts* a *positive* quantity; the rogue *adds* a *negative* quantity.

Again, suppose I have \$1000 in my possession, but owe \$400; it is immaterial to me whether a friend pays the debt of \$400 or gives me \$400; for in either case I shall be worth \$1000. In the former case the friend *subtracts* a *negative* quantity; in the latter, he *adds* a *positive*. Or, to make the proof general:

1st.	Suppose $+b$ to be taken from	$a+b$
	the result will be	a;
	and adding $-b$ to $a+b$ we have	$a+b-b$,
	which is, as before, equal to	a.
2d.	Suppose $-b$ to be taken from	$a-b$
	the result will be	a;
	and adding $+b$ to $a-b$ we have	$a-b+b$,
	which is, as before, equal to	a.

3. Subtract $b+c$ from a.

OPERATION.

$$a-(b+c)=a-b-c$$

b subtracted from a gives $a-b$; but in subtracting b we have subtracted too small a quantity by c, and therefore the remainder is too great by c, and the remainder sought is $a-b-c$.

4. Subtract $b-c$ from a.

OPERATION.

$$a-(b-c)=a-b+c$$

In subtracting b from a we subtract a quantity too great by c; therefore the remainder $(a-b)$ would be just so much too small, and the remainder sought is $a-b+c$.

43. By examining the examples just given it will be seen that in every case the sign of each term of the subtrahend is changed, and that the subsequent process is precisely the same as in addition; hence, for subtraction in Algebra we have the following

RULE.

Change the sign of each term of the subtrahend from + to —, or — to +, or suppose each to be changed, and then proceed as in addition.

	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)
Min.	9	9	9	9	9	9	9
Sub.	9	6	3	0	—3	—6	—9
Rem.	0	3	6	9	12	15	18

In examples 1–7, the minuend remaining the same while the subtrahend becomes in each 3 less, the remainder in each is 3 greater than in the preceding.

	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
Min.	9	6	3	0	—3	—6	—9
Sub.	9	9	9	9	9	9	9
Rem.	0	—3	—6	—9	—12	—15	—18

In examples 8–14, the minuend in each becoming 3 less while the subtrahend remains the same, the remainder in each is 3 less than in the preceding.

	(15.)	(16.)	(17.)	(18.)	(19.)	(20.)	(21.)
Min.	9	6	3	0	—3	—6	—9
Sub.	9	6	3	0	—3	—6	—9
Rem.	0	0	0	0	0	0	0

In examples 15–21, both minuend and subtrahend decreasing by 3, the remainder remains the same.

	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)
Min.	$26x$	$27axy$	$-13ab$	$-18c$	$49xy$	$-438b$
Sub.	$10x$	$-4axy$	$4ab$	$-6c$	$-25xy$	$27b$
Rem.	$16x$	$31axy$	$-17ab$	$-12c$		

	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)
Min.	$10x$	$-4axy$	$4ab$	$-6c$	$-25xy$	$27b$
Sub.	$26x$	$27axy$	$-13ab$	$-18c$	$49xy$	$-438b$
Rem.	$-16x$	$-31axy$	$17ab$	$12c$		

	(13.)	(14.)
Min.	$6x - 14y + 3z$	$7a + 18b - 10c$
Sub.	$3x + 3y + z$	$-25b + 6c - 8d$
Rem.	$3x - 17y + 2z$	$7a + 43b - 16c + 8d$

15. From $28x$ take $-17x$. Ans. $45x$.

16. From $-347a$ take $223a$. Ans. $-570a$.

17. From $-76y$ take $-33y$. Ans. $-43y$.

18. From $44b$ take $-150b$. Ans. $194b$.

19. From $-417c$ take $984c$.

20. From $-84z$ take $-117z$.

21. From $17ax - 18bc + 44xy$ take $25bc - 14xy + 20a$. Ans. $17ax - 43bc + 58xy - 20a$.

22. From $384x - 74y + 18c$ take $118x + 74y - 27c$. Ans. $266x - 148y + 45c$.

23. From $x^2 - y^3 + x^4 - 10x^3$ take $x^2 + 4y^3 - x^4 - 4x^3$. Ans. $2x^4 - 6x^3 - 5y^3$.

24. From $6aby - 4xy + 3xz$ take $-4aby - 3xz - 4xy$.

25. From $x^2 + 2xy + y^2$ take $x^2 - 2xy + y^2$.

26. From $x^2 + 2xy + y^2$ take $-x^2 + 2xy - y^2$.

44. The subtraction of a polynomial may be indicated by enclosing the polynomial in a parenthesis and prefixing the sign $-$.

Thus, $x^3 + y^3 - z^3$ taken from $x^3 - z^3$ may be written $x^3 - z^3 - (x^3 + y^3 - z^3)$.

When a parenthesis with the sign minus before it is removed, the sign of each term within the parenthesis must be changed according to the Rule for subtraction.

Thus, $x^3 - z^3 - (x^3 + y^3 - z^3) = x^3 - z^3 - x^3 - y^3 + z^3 = -y^3$.

And conversely,

A polynomial, or any number of the terms of a polynomial, can be enclosed in a parenthesis and the minus sign placed before the parenthesis without changing the value of the expression, providing the signs of all the terms are changed from plus to minus or from minus to plus.

Thus, $a^2 - b^2 + c^2 + d - x = a^2 - (b^2 - c^2 - d + x)$.

NOTE.—When the sign of the first term in the parenthesis is plus, the sign need not be written. (Art. 18.)

According to this principle a polynomial can be written in a variety of ways.

$$
\begin{aligned}
\text{Thus, } x^3 - 3x^2y + 3xy^2 - y^3 &= x^3 - (3x^2y - 3xy^2 + y^3) \\
&= x^3 - 3x^2y - (-3xy^2 + y^3) \\
&= x^3 + 3xy^2 - (3x^2y + y^3) \\
&= x^3 - y^3 - (3x^2y - 3xy^2) \text{ \&c.}
\end{aligned}
$$

Remove the parenthesis, and reduce each of the following examples to its simplest form.

1. $a^2 - (2ab + c^2)$. Ans. $a^2 - 2ab - c^2$.
2. $x^2 - 6ax + x^3 - 6x^2y - (x^2 + 6ax + x^3 - 6x^2y)$. Ans. $-12ax$.
3. $m^2 - n^2 + 2x - (4m^2 + 3n^2 - 4c)$.
4. $16xy + 14c - 18y - (-14c + 27y - 16xy)$. Ans. $32xy + 28c - 45y$.
5. $4x^2y - (8xy^2 - 7x^2y^2 + 8x^3y)$.
6. $-(-x^2 + 7 - 25xy + y^3)$.

Place in parenthesis, with the sign — prefixed, without changing the value of the expression,

1. The last three terms of $7x^2 - 14xy - 3z + 4y$.
Ans. $7x^2 - (14xy + 3z - 4y)$.

2. The last three terms of $x^2 + y^2 - 3xy + 4c$.
Ans. $x^2 - (3xy - y^2 - 4c)$.

3. The last four terms of $4a - 7b - 6c - 8d + x^2$.

4. The last four terms of $a^2 + b^2 + c^2 - d^2 + a^3$.

5. Write in as many forms as possible by enclosing two or more of the terms in parenthesis, $a^3 - b^3 + c^3 - d^3$.

45. In subtraction, when two quantities have a common factor their difference is the difference of the coefficients of the common factor multiplied by this factor.

Thus, $ax - bx = (a - b)x$.

1. From ax^2 take $cx^2 - dx^2$. Ans. $(a - c + d)x^2$.

2. From $4\sqrt{x}$ take $a\sqrt{x} + b\sqrt{x}$.
Ans. $(4 - a - b)\sqrt{x}$.

3. From ax^3 take $bx^3 - bx^2$. Ans. $(a - b)x^3 + bx^2$.

4. From $4x^2 - 6x$ take $ax^2 + bx$.
Ans. $(4 - a)x^2 - (6 + b)x$.

5. From $6a^3 + 4a^2 - a$ take $a^3x - a^2y + az$.
Ans. $(6 - x)a^3 + (4 + y)a^2 - (1 + z)a$.

6. From $ab - bc$ take $3b + cx$.

7. From $a^2 - bx + c\sqrt{x}$ take $bx^2 + cx - d\sqrt{x}$.

8. From $xy^2 + x^2 - x^2y^2$ take $y^2 + x^2y - x^2y^2$.

SECTION VI.

MULTIPLICATION.

46. MULTIPLICATION is a short method of finding the sum of the repetitions of a quantity.

47. The multiplier must always be an abstract number, and the product is always of the *same nature* as the multiplicand.

The cost of 4 pounds of sugar at 17 cents a pound is 17 cents taken, not 4 pounds times, but 4 times; and the product is of the same denomination as the multiplicand 17, viz. cents.

In Algebra the sign of the multiplier shows whether the repetitions are to be added or subtracted.

1. $(+a) \times (+4) = +4a$;
i. e. $+a$ added 4 times is $+a+a+a+a=+4a$.

2. $(+a) \times (-4) = -4a$;
i. e. $+a$ subtracted 4 times is $-a-a-a-a=-4a$.

3. $(-a) \times (+4) = -4a$;
i. e. $-a$ added 4 times is $-a-a-a-a=-4a$.

4. $(-a) \times (-4) = +4a$;
i. e. $-a$ subtracted 4 times is $+a+a+a+a=+4a$.

In the first and second examples the *nature* of the product is $+$; in the first, the $+$ sign of 4 shows that the product is to be added, and $+4a$ added is $+4a$; in the second, the $-$ sign of 4 shows that the product is to be subtracted, and $+4a$ subtracted is $-4a$. In the third and fourth examples the *nature* of the product is $-$; in the third, the $+$ sign of 4 shows that the product is to be added, and $-4a$ added is $-4a$; in the

fourth, the $-$ sign of 4 shows that the product is to be subtracted, and $-4a$ subtracted is $+4a$.

48. Hence in multiplication we have for the sign of the product the following

RULE.

Like signs give $+$; *unlike,* $-$.

Hence the products of an *even* number of negative factors is positive, of an *odd* number, negative.

49. Multiplication in Algebra can be presented best under three cases.

CASE I.

50. When both factors are monomials.

1. Multiply $3a$ by $2b$.

OPERATION.

$$3a \times 2b = 3 \times a \times 2 \times b = 3 \times 2 \times a \times b = 6ab.$$

As the product is the same in whatever order the factors are arranged, we have simply changed their order and united in one product the numerical coefficients.

Hence, when both factors are monomials,

RULE.

Annex the product of the literal factors to the product of their coefficients, remembering that like signs give $+$ *and unlike,* $-$.

2. Multiply a^3 by a^2.

OPERATION.

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a \times a \times a \times a \times a = a^5$$

As the exponent of a quantity shows how many times it is taken as a factor, $a^3 = a \times a \times a$; and $a^2 = a \times a$; and $a^3 \times a^2 = a \times a \times a \times a \times a$, and this is equal to a^5. (Art. 24.) Hence,

Powers of the same quantity are multiplied together by adding their exponents.

(3.)	(4.)	(5.)	(6.)	(7.)
$4xy$	$5x^2y^2$	$7ab$	$-14mn^3$	$-a^2b^4$
$3ab$	$3xy^3$	$-8a^2b$	$6an^4$	$-4a^2b$
$12abxy$	$15x^3y^5$	$-56a^3b^2$	$-84amn^7$	$4a^4b^5$

8. Multiply x^2 by x^3.
9. Multiply x^4 by x^7.
10. Multiply a^3 by $-a$.
11. Multiply $-a^4$ by a^6.
12. Multiply $-c^8$ by $-c^4$.
13. Multiply $8xy^2$ by $7ax$.
14. Multiply $504a^2b^3$ by $-8a^3b$.
15. Multiply $-25xyz$ by $4xyz$.
16. Multiply $-417abc^2$ by $-3ab^2c$.
17. Multiply together $444xy$, $3x^2y^3$, and $-2z$.
    Ans. $-2664x^3y^4z$.
18. Multiply $4a^2b^2cd$ by $-4abc^2d^2$.
19. Multiply $-5x^m$ by $-6x^n$. Ans. $30x^{m+n}$.
20. Multiply together $14abc^2$, $-5a^2bc$, and $-4ab^2$
21. Multiply $25\sqrt{ay}$ by $3bx$. Ans. $75bx\sqrt{ay}$.
22. Multiply $4(x+y)$ by $3(x+y)$.
    Ans. $12(x+y)^2$.

NOTE. — Any number of terms enclosed in a parenthesis may be treated as a monomial.

23. Multiply $-12(a^2-b^2)$ by $-4(a^2-b^2)$.
    Ans. $48(a^2-b^2)^2$.
24. Multiply $(a-x)^4$ by $(a-x)^2$.
25. Multiply $4(a+b)^m$ by $2(a+b)^n$.
    Ans. $8(a+b)^{m+n}$.
26. Multiply $a^3(x+z)^2$ by $ab^2(x+z)$.

CASE II.

51. When only one factor is a monomial.

1. Multiply $8 + 5$ by 3.

OPERATION.

$$\left.\begin{array}{r} 8 + 5 \\ 8 + 5 \\ 8 + 5 \\ \hline 24 + 15 \end{array}\right\} \text{ or } \begin{array}{r} 8 + 5 = 13 \\ 3 \quad\;\; 3 \\ \hline 24 + 15 = 39 \end{array}$$

In this example, not the sum of 3 repetitions of 8 *only*, but of 8 *and* 5, is required; the sum of 3 repetitions of $8 = 24$; of 3 repetitions of $5 = 15$. Hence, the sum of 3 repetitions of $8 + 5 = 24 + 15$.

2. Multiply $8 - 5$ by 3.

OPERATION.

$$\left.\begin{array}{r} 8 - 5 \\ 8 - 5 \\ 8 - 5 \\ \hline 24 - 15 \end{array}\right\} \text{ or } \begin{array}{r} 8 - 5 = 3 \\ 3 \quad\;\; 3 \\ \hline 24 - 15 = 9 \end{array}$$

The sum of 3 repetitions of $8 = 24$; but it is not the sum of 3 repetitions of 8 that is required, but of a number 5 units less than 8; 24, therefore, will have in it the sum of 5 units repeated 3 times, or 15, too much; the product required, therefore, is $24 - 15$.

Therefore,

The product of the sum is equal to the sum of the products, and the product of the difference to the difference of the products.

3. Multiply $x + y - z$ by a.

OPERATION.

$$\begin{array}{r} x + y - z \\ a \\ \hline ax + ay - az \end{array}$$

The sum of the repetitions of x a times, of y a times, and of $-z$ a times is $ax + ay - az$.

RULE.

Multiply each term of the multiplicand by the multiplier, and connect the several results by their proper signs.

(4.)	(5.)
$4x^2 - 8x + 14y$	$a^2 - 2ab + b^2$
$3ax$	$-4x$
$12ax^3 - 24ax^2 + 42axy$	$-4a^2x + 8abx - 4b^2x.$

6. Multiply $5mn + 4m^2 - 6n^2$ by $4ab$.
7. Multiply $16a^2x - 8xz + 4y$ by $-3xy$.
8. Multiply $bx^3 - cx^2 + dx$ by $-x^3$.
9. Multiply $-63xy - 14x - 6z$ by $-4z$.
   Ans. $252xyz + 56xz + 24z^2$.
10. Multiply $14a^4 - 13a^3 + 12a^2 - 11a$ by $4a^5$.
11. Multiply $x - 2a + 14$ by ax.
12. Multiply $17ax - 14by + 11cz$ by $-4abcxyz$.
13. Multiply $21a^2b^2 - 3xy^2 - 4bc$ by $-9axy$.

CASE III.

52. When both factors are polynomials.

1. Multiply $7 + 4$ by $5 - 3$.

OPERATION.

$$\begin{array}{ll} 7+4 & = 11 \\ 5-3 & = 2 \\ \hline 35+20-21-12 & = 22 \end{array}$$

Multiplying $7 + 4$ by 5 is taking the multiplicand 3 too many times; therefore, the true product will be found by subtracting $3(7 + 4)$ from $5(7 + 4)$.

2. Multiply $x - y$ by $a + b$.

OPERATION.

$$\begin{array}{l} x-y \\ a+b \\ \hline ax-ay+bx-by \end{array}$$

a times $x - y = ax - ay$; but $x - y$ is to be taken, not a times only, but $a + b$ times; therefore, $a(x - y)$ is too small by $b(x - y)$; and the product required is $ax - ay + bx - by$. Hence,

RULE.

Multiply each term of the multiplicand by each term of the multiplier, and find the sum of the several products.

(3.)

$$\begin{array}{r}
a^2 + 2\,a\,x + x^2 \\
a - x \\
\hline
a^3 + 2\,a^2 x + \quad a\,x^2 \qquad\quad \\
- \quad a^2 x - 2\,a\,x^2 - x^3 \\
\hline
a^3 + \quad a^2 x - \quad a\,x^2 - x^3
\end{array}$$

(4.)

$$\begin{array}{l}
x\,y + a\,b \\
x\,y - a\,b \\
\hline
x^2 y^2 + a\,b\,x\,y \\
\qquad\quad - a\,b\,x\,y - a^2 b^2 \\
\hline
x^2 y^2 - a^2 b^2
\end{array}$$

5. Multiply $a^2 + b^2 - c^2$ by $a^2 + c^2$.
Ans. $a^4 + a^2 b^2 + b^2 c^2 - c^4$.

6. Multiply $x^2 - 2\,x\,y + y^2$ by $x^2 + y^2$.
Ans. $x^4 - 2\,x^3 y + 2\,x^2 y^2 - 2\,x\,y^3 + y^4$.

7. Multiply $4\,a^4 - 2\,a^3 b + 3\,a^2 b^2$ by $2\,a^2 - 2\,b^2$.
Ans. $8\,a^6 - 4\,a^5 b - 2\,a^4 b^2 + 4\,a^3 b^3 - 6\,a^2 b^4$.

8. Multiply $x^4 + 2\,x^3 + 3\,x^2 + 2\,x + 1$ by $x^2 - 2\,x + 1$.

9. Multiply $x^2 + y^2 + z^2 - x\,y - x\,z - y\,z$ by $x + y + z$.
Ans. $x^3 + y^3 + z^3 - 3\,x\,y\,z$.

10. Multiply $4\,x^5 - 7\,x^4 + 10\,x^3 - 13\,x^2$ by $3\,x - 2$.

11. Multiply $a^3 + a^2 - 1$ by $a^2 - 1$.

12. Multiply $x^2 + 7\,a\,x - 14\,a^3$ by $x - 7\,a$.
Ans. $x^3 - 49\,a^2 x - 14\,a^3 x + 98\,a^4$.

13. Multiply $x + y - a$ by $x - y + a$.

14. Multiply $a^n + b^n$ by $a^m - b^m$.
Ans. $a^{m+n} + a^m b^n - a^n b^m - b^{m+n}$.

15. Multiply $7\,x\,y - 14\,x^2 y^2 + 21\,x^3 y^3$ by $6\,x\,y - 3$.
Ans. $-21\,x\,y + 84\,x^2 y^2 - 147\,x^3 y^3 + 126\,x^4 y^4$.

16. Multiply $6\,a^2 b - 9\,a\,b^2 - 12\,a^2 b^2$ by $2\,a\,b - 3\,b^2$.
Ans. $12\,a^3 b^2 - 36\,a^2 b^3 - 24\,a^3 b^3 + 27\,a\,b^4 + 36\,a^2 b^4$.

17. Multiply $x^4 - x^3 + x^2 - x + 1$ by $x + 1$.
Ans. $x^5 + 1$.

18. Multiply $x^4 - x^3 + x^2 - x + 1$ by $x - 1$.
Ans. $x^5 - 2\,x^4 + 2\,x^3 - 2\,x^2 + 2\,x - 1$.

19. Multiply $x^4 + x^3 + x^2 + x + 1$ by $x + 1$.

20. Multiply $x^4 + x^3 + x^2 + x + 1$ by $x - 1$.

SECTION VII.

DIVISION.

53. DIVISION is finding a quotient which, multiplied by the divisor, will produce the dividend.

In accordance with this definition and the Rule in Art. 48, the sign of the quotient must be + when the divisor and the dividend have like signs; — when the divisor and the dividend have unlike signs; i. e. in division as in multiplication we have for the signs the following

RULE.

Like signs give +; unlike, —.

CASE I.

54. When the divisor and dividend are both monomials.

1. Divide $6\,a\,b$ by $2\,b$.

OPERATION.

$$6\,a\,b \div 2\,b = 3\,a$$

The coefficient of the quotient must be a number which, multiplied by 2, the coefficient of the divisor, will give 6, the coefficient of the dividend; i. e. 3: and the literal part of the quotient must be a quantity which, multiplied by b, will give $a\,b$; i. e. a: the quotient required, therefore, is $3\,a$.

Hence, for division of monomials,

RULE.

Annex the quotient of the literal quantities to the quotient of their coefficients, remembering that like signs give + and unlike, —.

2. Divide a^5 by a^2.

OPERATION.

$a^5 \div a^2 = a^3$ For $a^3 \times a^2 = a^5$. (Art. 50.) Hence,

Powers of the same quantity are divided by each other by subtracting the exponent of the divisor from that of the dividend.

(3.)
$$\frac{27\,x^2\,y^2}{9\,x\,y} = 3\,x\,y$$

(4.)
$$\frac{48\,a^2\,x\,y}{-16\,a\,x} = -3\,a\,y$$

(5.)
$$\frac{-276\,x^3\,y}{46\,x^2\,y} = -6\,x$$

(6.)
$$\frac{-16\,a^2\,x^2\,y\,z}{-4\,a\,x\,y\,z} = 4\,a\,x$$

7. Divide $34\,a\,b\,y^2$ by $2\,a\,y$.
8. Divide $297\,x^2\,y^3$ by $-99\,x\,y^2$. Ans. $-3\,x\,y$.
9. Divide $-74\,x\,y^2\,z$ by $2\,x\,y$.
10. Divide $-144\,a^2\,b^2\,x$ by $-24\,a\,b\,x$. Ans. $6\,a\,b$.
11. Divide $a\,x^4$ by $a\,x^3$.
12. Divide $8\,x^6$ by $-8\,x^4$.
13. Divide $-210\,x^7\,y$ by $42\,x^6\,y$.
14. Divide $-270\,a\,b\,x$ by $-135\,a\,b\,x$.
15. Divide $-474\,a^3\,b^3\,c^2$ by $158\,a\,b^3\,c$.
16. Divide x^m by x^n. Ans. x^{m-n}.
17. Divide $14\,a^m\,x^n$ by $-7\,a^n\,x$. Ans. $-2\,a^{m-n}\,x^{n-1}$.
18. Divide $-747\,a^3\,b^4\,c^5\,d^6$ by $83\,a^2\,b\,c\,d^6$.
19. Divide $12\,(x+y)^2$ by $4\,(x+y)$. Ans. $3\,(x+y)$.
20. Divide $-27\,(a-b)^4$ by $9\,(a-b)^2$.
21. Divide $(b-c)^5$ by $(b-c)^2$. Ans. $(b-c)^3$.
22. Divide $14\,(x-y)^7$ by $7\,(x-y)^2$.

CASE II.

55. When the divisor only is a monomial.

1. Divide $a x + a y + a z$ by a.

OPERATION.

$$a) \underline{a x + a y + a z}$$
$$x + y + z$$

In the multiplication of a polynomial by a monomial, each term of the multiplicand is multiplied by the multiplier; and therefore we divide each term of the dividend $a x + a y + a z$ by the divisor a, and connect the partial quotients by their proper signs. Hence,

RULE.

Divide each term of the dividend by the divisor, and connect the several results by their proper signs.

(2.)

$$3 a) \underline{6 a x^2 - 24 a x^3}$$
$$2 x^2 - 8 x^3$$

(3.)

$$-5 x^2 y) \underline{-15 x^3 y - 25 x^2 y}$$
$$3 x + 5$$

4. Divide $12 a x^3 - 24 a x^2 + 42 a x y$ by $3 a x$.

5. Divide $-4 a^2 x + 8 a b x - 4 b^2 x$ by $-4 x$.

6. Divide $6 a^2 x^4 - 12 a^3 x^3 + 15 a^4 x^5$ by $3 a^2 x^2$.
Ans. $2 x^2 - 4 a x + 5 a^2 x^3$.

7. Divide $12 a^4 y^6 - 16 a^5 y^5 + 20 a^6 y^4 - 28 a^7 y^3$ by $4 a^4 y^3$.

8. Divide $-5 x^3 + 10 x^2 - 15 x$ by $-5 x$.

9. Divide $273 (a + x)^2 - 91 (a + x)$ by $91 (a + x)$.
Ans. $3 (a + x) - 1 = 3 a + 3 x - 1$.

10. Divide $20 a b c - 4 a c + 8 a c d - 12 a^2 c^2$ by $-4 a c$.

11. Divide $16 a^2 x^2 - 32 a^3 x^3 + 48 a^4 x^4$ by $16 a^2 x^2$.

12. Divide $72 x^2 y^2 - 36 x^3 y^3 - 54 x^3 y^2 z^3$ by $-18 x^2 y^2$.
Ans. $-4 + 2 x y + 3 x z^3$.

13. Divide $18 a x^3 y - 54 x^2 y^2 + 108 c x^5 y^3$ by $9 x^2 y$.

14. Divide $40 a^4 b^2 + 8 a^2 b^2 - 96 a^8 b^8 x^3$ by $8 a^2 b^2$.

15. Divide $39 x^4 z^5 - 65 a x^3 z^6 + 13 x^3 z^5$ by $-13 x^3 z^5$.

CASE III.

56. When the divisor and dividend are both polynomials.

1. Divide $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.

OPERATION.

$$
\begin{array}{l}
x^2 - 2xy + y^2)\ x^3 - 3x^2y + 3xy^2 - y^3\ (x - y \\
\qquad\qquad\qquad\ \ \underline{x^3 - 2x^2y + \ \ xy^2} \\
\qquad\qquad\qquad\qquad\ \ - \ \ x^2y + 2xy^2 - y^3 \\
\qquad\qquad\qquad\qquad\ \ \underline{- \ \ x^2y + 2xy^2 - y^3}
\end{array}
$$

The divisor and dividend are arranged in the order of the powers of x, beginning with the highest power. x^3, the highest power of x in the dividend, must be the product of the highest power of x in the quotient and x^2 in the divisor; therefore, $\frac{x^3}{x^2} = x$ must be the highest power of x in the quotient. The divisor $x^2 - 2xy + y^2$ multiplied by x must give several of the partial products which would be produced were the divisor multiplied by the whole quotient. When $(x^2 - 2xy + y^2)\,x = x^3 - 2x^2y + xy^2$ is subtracted from the dividend, the remainder must be the product of the divisor and the remaining terms of the quotient; therefore we treat the remainder as a new dividend, and so continue until the dividend is exhausted.

Hence, for the division of polynomials we have the following

RULE.

Arrange the divisor and dividend in the order of the powers of one of the letters.

Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient.

Multiply the whole divisor by this quotient, and subtract the product from the dividend.

Consider the remainder as a new dividend, and proceed as before until the dividend is exhausted.

NOTE.—If the dividend is not exactly divisible by the divisor, the remainder must be placed over the divisor in the form of a fraction and connected with the quotient by the proper sign.

2. Divide $x^4 + 4y^4$ by $x^2 - 2xy + 2y^2$.

$$
\begin{array}{l}
x^2 - 2xy + 2y^2)\ x^4 + 4y^4 \qquad (x^2 + 2xy + 2y^2 \\
\qquad\qquad x^4 - 2x^3y + 2x^2y^2 \\
\hline
\qquad\qquad\qquad 2x^3y - 2x^2y^2 + 4y^4 \\
\qquad\qquad\qquad 2x^3y - 4x^2y^2 + 4xy^3 \\
\hline
\qquad\qquad\qquad\qquad 2x^2y^2 - 4xy^3 + 4y^4 \\
\qquad\qquad\qquad\qquad 2x^2y^2 - 4xy^3 + 4y^4 \\
\hline
\end{array}
$$

NOTE.—By multiplying the quotient and divisor together all the terms which appear in the process of dividing will be found in the partial products.

3. Divide $x^4 - 1$ by $x - 1$.

$$
\begin{array}{l}
x - 1)\ x^4 - 1\ (x^3 + x^2 + x + 1 \\
\qquad\ x^4 - x^3 \\
\hline
\qquad\qquad x^3 - 1 \\
\qquad\qquad x^3 - x^2 \\
\hline
\qquad\qquad\quad x^2 - 1 \\
\qquad\qquad\quad x^2 - x \\
\hline
\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad x - 1 \\
\hline
\end{array}
$$

4. Divide $ax - ay + bx - by + z$ by $x - y$.

$$
\begin{array}{l}
x - y)\ ax - ay + bx - by + z\ (a + b + \dfrac{z}{x - y} \\
\qquad ax - ay \\
\hline
\qquad\qquad bx - by \\
\qquad\qquad bx - by \\
\hline
\qquad\qquad\qquad z
\end{array}
$$

5. Divide $2by - 2b^2y - 3b^2yz + 6b^3y + byz - yz$ by $2b - z$. Ans. $3b^2y - by + y$.

6. Divide $c^3 + x^3$ by $c + x$. Ans. $c^2 - cx + x^2$.

7. Divide $a^3 + a^2 + a^2x + ax + 3ac + 3c$ by $a + 1$.
Ans. $a^2 + ax + 3c$.

8. Divide $a + b - d - ax - bx + dx$ by $a + b - d$.

9. Divide $2a^4 - 13a^3y + 11a^2y^2 - 8ay^3 + 2y^4$ by $2a^2 - ay + y^2$. Ans. $a^2 - 6ay + 2y^2$.

10. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

11. Divide $2x^3 - 19x^2 + 26x - 19$ by $x - 8$.
Ans. $2x^2 - 3x + 2 - \frac{3}{x - 8}$.

12. Divide $x^5 + 1$ by $x + 1$.
Ans. $x^4 - x^3 + x^2 - x + 1$.

13. Divide $x^6 - 1$ by $x - 1$.
Ans. $x^5 + x^4 + x^3 + x^2 + x + 1$.

14. Divide $x^6 - 1$ by $x + 1$.

15. Divide $x^5 - y^5$ by $x - y$.
Ans. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

16. Divide $a^3 - x^3$ by $a - x$.

17. Divide $m^6 - n^6$ by $m^2 + mn + n^2$.
Ans. $m^4 - m^3n + mn^3 - n^4$.

18. Divide $4a^4 - 9a^2 + 6a - 3$ by $2a^2 + 3a - 1$.

19. Divide $x^4 + 4x^2y^2 + 3y^4$ by $x + 2y$.

20. Divide $a^4 - a^2x^2 + 2ax^3 - x^4$ by $a^2 - ax + x^2$.

21. Divide $x^6 + 2x^3y^3 + y^6$ by $x^2 - xy + y^2$.

22. Divide $1 - a^4$ by $1 + a + a^2 + a^3$.
Ans. $1 - a$.

23. Divide $10x^3 - 20x^2y + 30y^3$ by $x + y$.

24. Divide $7ax^4 + 21ax^3 + 14a$ by $x + 1$.
Ans. $7ax^3 + 14ax^2 - 14ax + 14a$.

25. Divide $27a^8y^4 - 8a^6y$ by $3y^2 - 2ay$.

SECTION VIII.

DEMONSTRATION OF THEOREMS.

57. From the principles already established we are prepared to demonstrate the following theorems.

THEOREM I.

The sum of two quantities plus their difference is twice the greater; and the sum of two quantities minus their difference is twice the less.

Let a and b represent the two quantities, and $a > b$; their sum is $a + b$; their difference, $a - b$.

PROOF.

1st. $(a + b) + (a - b) = a + b + a - b = 2a$;

2d. $(a + b) - (a - b) = a + b - a + b = 2b$.

Therefore, when the sum and difference of two quantities are given to find the quantities,

RULE.

Subtract the difference from the sum, and divide the remainder by two, and we shall have the less; the less plus the difference will be the greater.

In the following examples the sum and difference are given and the quantities required.

1. 16 and 12. Ans. 14 and 2.
2. 272 and 18.
3. 456 and 84.
4. Sum $2x$ and difference $2y$.
   Ans. $x + y$ and $x - y$.
5. Sum $7x^2 + 3y$ and difference $5x^2 - 3y$.
6. Sum $2a - 8b$ and difference $10a + 14b$.
   Ans. $6a + 3b$ and $-4a - 11b$.

THEOREM II.

58. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

Let a and b represent the two quantities; their sum will be $a + b$.

PROOF.

$$\begin{array}{l}
a + b \\
a + b \\
\hline
a^2 + ab \\
\quad\; + ab + b^2 \\
\hline
a^2 + 2ab + b^2
\end{array}$$

According to this theorem, find the square of

1. $x + y$. Ans. $x^2 + 2xy + y^2$.
2. $2x + 2y$. Ans. $4x^2 + 8xy + 4y^2$.
3. $x + 1$.
4. $4 + x$.
5. $2x + 3y$. Ans. $4x^2 + 12xy + 9y^2$.
6. $3a + b$.

THEOREM III.

59. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

Let a and b represent the two quantities, and $a > b$; their difference will be $a - b$.

PROOF.

$$\begin{array}{l}
a - b \\
a - b \\
\hline
a^2 - ab \\
\quad\; - ab + b^2 \\
\hline
a^2 - 2ab + b^2
\end{array}$$

According to this theorem, find the square of

1. $x - y$. Ans. $x^2 - 2xy + y^2$.

2. $2x - 4y$.

3. $x - 1$. Ans. $x^2 - 2x + 1$.

4. $7x - 2$.

THEOREM IV.

60. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

Let $a + b$ be the sum, and $a - b$ the difference of the two quantities a and b.

PROOF.

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \quad\; - ab - b^2 \\ \hline a^2 \qquad\quad - b^2 \end{array}$$

According to this theorem, multiply

1. $x + y$ by $x - y$. Ans. $x^2 - y^2$.

2. $2x + 1$ by $2x - 1$.

3. $x^2 + y^2$ by $x^2 - y^2$. Ans. $x^4 - y^4$.

4. $3x + 4$ by $3x - 4$.

5. $3xy + 4ab$ by $3xy - 4ab$.

61. This theorem suggests an easy method of squaring numbers. For, since $a^2 = (a - b)(a + b) + b^2$,

$$99^2 = (99 - 1)(99 + 1) + 1^2 = 98 \times 100 + 1 = 9801.$$

In like manner,

$$96^2 = 92 \times 100 + 16 = 9216.$$
$$998^2 = 996 \times 1000 + 4 = 996004.$$
$$497^2 = 494 \times 500 + 9 = 247 \times 1000 + 9 = 247009.$$

In accordance with this principle find the square of

1. 98.	4. 493.	7. 888.
2. 89.	5. 789.	8. 999.
3. 45.	6. 698.	9. 1104.

Miscellaneous Examples.

1. Find the square of $3x - 6y$.
Ans. $9x^2 - 36xy + 36y^2$.

2. Find the square of $4axy + 7abx$.
Ans. $16a^2x^2y^2 + 56a^2bx^2y + 49a^2b^2x^2$.

3. Multiply $7x + 1$ by $7x - 1$. Ans. $49x^2 - 1$.

4. Required those two quantities whose sum is $3x + 2a$ and difference $x - 2a$. Ans. $2x$ and $x + 2a$.

5. Expand $(x^2 - 4)^2$.

6. Multiply $4ab + 3$ by $4ab - 3$.

7. Find the square of $14a^2b^2 + 10x^2y$.

8. Find the square of $4a - b$.

9. Multiply $10x + 2$ by $10x - 2$.

10. Find the square of $3ax - 8axy$.
Ans. $9a^2x^2 - 48a^2x^2y + 64a^2x^2y^2$.

11. Find the square of $2a + b$.

12. Find the value of $(6a + 4)(6a - 4)(36a^2 + 16)$.
Ans. $1296a^4 - 256$.

13. Find the square of $10a^2 - 5b^2$.

14. Expand $(3a^2x + 4by^3)^2$.
Ans. $9a^4x^2 + 24a^2bxy^3 + 16b^2y^6$.

15. Find the product of $a^{16} + 1$, $a^8 + 1$, $a^4 + 1$, $a^2 + 1$, $a + 1$, and $a - 1$. Ans. $a^{32} - 1$.

16. Find the product of $a + b$, $a - b$, and $a^2 - b^2$.

SECTION IX.

FACTORING.

62. Factoring is the resolving a quantity into its factors.

63. The factors of a quantity are those integral quantities whose continued product is the quantity.

Note.—In using the word factor we shall exclude unity.

64. A Prime Quantity is one that is divisible without remainder by no integral quantity except itself and unity.

Two quantities are mutually prime when they have no common factor.

65. The Prime Factors of a quantity are those prime quantities whose continued product is the quantity.

66. The factors of a purely algebraic monomial quantity are apparent. Thus, the factors of a^2bxyz are $a \times a \times b \times x \times y \times z$.

67. Polynomials are factored by inspection, in accordance with the principles of division and the theorems of the preceding section.

CASE I.

68. When all the terms have a common factor.

1. Find the factors of $ax - ab + ac$.

OPERATION.

$$(ax - ab + ac) = a(x - b + c)$$

As a is a factor of each term it must be a factor of the polynomial; and if we divide the polynomial by a, we obtain the other factor. Hence,

RULE.

Write the quotient of the polynomial divided by the common factor in a parenthesis, with the common factor prefixed as a coefficient.

2. Find the factors of $6xy - 72xy^2 + 18ax^2y^3$.
Ans. $6xy(1 - 12y + 3axy^2)$.

NOTE.—*Any* factor common to all the terms can be taken as well as $6xy$; 2, 3, x, y, or the product of any two or more of these quantities, according to the result which is desired. In the examples given, let the greatest monomial factor be taken.

3. Find the factors of $x + x^2$. Ans. $x(1 + x)$.

4. Find the factors of $8a^2x^2 + 12a^3x^4 - 4axy$.
Ans. $4ax(2ax + 3a^2x^3 - y)$.

5. Find the factors of $5x^4y^2 + 25ax^5 - 15x^3y^3$.
Ans. $5x^3(xy^2 + 5ax^2 - 3y^3)$.

6. Find the factors of $7ax - 8by + 14x^2$.

7. Find the factors of $4x^2y^2 - 28x^3y^4 - 44x^4y^2$.

8. Find the factors of $55a^2c - 11ac + 33a^2cx$.

9. Find the factors of $98a^2x^2 - 294a^3x^2y^2$.

10. Find the factors of $15a^2b^2cd - 9ab^2d^2 + 18a^3c^2d^4$.

CASE II.

69. When two terms of a trinomial are perfect squares and positive, and the third term is equal to twice the product of their square roots.

1. Find the factors of $a^2 + 2ab + b^2$.

OPERATION.

$$a^2 + 2ab + b^2 = (a + b)(a + b)$$

We resolve this into its factors at once by the converse of the principle in Theorem II. Art. 58.

2. Find the factors of $a^2 - 2ab + b^2$.

OPERATION.

$a^2 - 2ab + b^2 = (a - b)(a - b)$

We resolve this into its factors at once by the converse of the principle in Theorem III. Art. 59. Hence,

RULE.

Omitting the term that is equal to twice the product of the square roots of the other two, take for each factor the square root of each of the other two connected by the sign of the term omitted.

3. Find the factors of $x^2 - 2xy + y^2$.
Ans. $(x - y)(x - y)$.

4. Find the factors of $4a^2c^2 + 12acd + 9d^2$.
Ans. $(2ac + 3d)(2ac + 3d)$.

5. Find the factors of $1 - 4xz + 4x^2z^2$.
Ans. $(1 - 2xz)(1 - 2xz)$.

6. Find the factors of $9x^2 - 6x + 1$.
Ans. $(3x - 1)(3x - 1)$.

7. Find the factors of $25x^2 + 60x + 36$.

8. Find the factors of $49a^2 - 14ax + x^2$.
Ans. $(7a - x)(7a - x)$.

9. Find the factors of $16y^2 - 16a^2y + 4a^4$.

10. Find the factors of $12ax + 4x^2 + 9a^2$.

11. Find the factors of $6x + 1 + 9x^2$.

CASE III.

70. When a binomial is the difference between two squares.

1. Find the factors of $a^2 - b^2$.

OPERATION.

$a^2 - b^2 = (a + b)(a - b)$

We resolve this into its factors at once by the converse of the principle in Theorem IV. Art. 60. Hence,

RULE.

Take for one of the factors the sum, and for the other the difference, of the square roots of the terms of the binomial.

2. Find the factors of $x^2 - y^2$.

Ans. $(x + y)(x - y)$.

3. Find the factors of $4a^2 - 9b^4$.

Ans. $(2a + 3b^2)(2a - 3b^2)$.

4. Find the factors of $16x^2 - c^2$.
5. Find the factors of $a^2b^4c^2 - x^2y^6$.
6. Find the factors of $81x^4 - 49y^2$.
7. Find the factors of $25a^2 - 4c^4$.
8. Find the factors of $m^8 - n^{10}$.

NOTE.—When the exponents of each term of the residual factor obtained by this rule are even, this factor can be resolved again by the same rule. Thus, $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$; but $x^2 - y^2 = (x + y)(x - y)$; and therefore the factors of $x^4 - y^4$ are $x^2 + y^2$, $x + y$, and $x - y$.

9. Find the factors of $a^4 - b^4$.

Ans. $(a^2 + b^2)(a + b)(a - b)$.

10. Find the factors of $x^8 - y^8$.

Ans. $(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$.

11. Find the factors of $a^4 - 1$.
12. Find the factors of $1 - x^8$.

Ans. $(1 + x^4)(1 + x^2)(1 + x)(1 - x)$.

13. Find the factors of $a^7 - a^5$.

Ans. $a^5(a + 1)(a - 1)$.

14. Find three factors of $x^9 - x^3$.

71. Any binomial consisting of the difference of the same powers of two quantities, or the sum of the same odd powers, can be factored. For

I. *The difference of the same powers of two quantities is divisible by the difference of the quantities.*

Let a and b represent two quantities and $a > b$, and by actual division we find

$$\frac{a^2 - b^2}{a - b} = a + b,$$

$$\frac{a^3 - b^3}{a - b} = a^2 + a\,b + b^2,$$

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2\,b + a\,b^2 + b^3,$$

and so on.

II. *The difference of the same even powers of two quantities is divisible by the sum of the quantities.*

$$\frac{a^2 - b^2}{a + b} = a - b,$$

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2\,b + a\,b^2 - b^3,$$

$$\frac{a^6 - b^6}{a + b} = a^5 - a^4\,b + a^3\,b^2 - a^2\,b^3 + a\,b^4 - b^5,$$

and so on.

It follows from the two preceding statements that

The difference of the same even powers of two quantities is divisible by either the sum or the difference of the quantities.

III. *The sum of the same odd powers of two quantities is divisible by the sum of the quantities.*

$$\frac{a^3 + b^3}{a + b} = a^2 - a\,b + b^2,$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3\,b + a^2\,b^2 - a\,b^3 + b^4,$$

$$\frac{a^7 + b^7}{a + b} = a^6 - a^5\,b + a^4\,b^2 - a^3\,b^3 + a^2\,b^4 - a\,b^5 + b^6,$$

and so on.

1. Find the factors of $x^5 - y^5$.

OPERATION.

$$(x^5 - y^5) \div (x - y) = x^4 + x^3 y + x^2 y^2 + x y^3 + y^4$$

By I. of this article, the difference of the same powers of two quantities is divisible by the difference of the quantities; therefore $x - y$ must be a factor of $x^5 - y^5$; and dividing $x^5 - y^5$ by $x - y$ gives the other factor $x^4 + x^3 y + x^2 y^2 + x y^3 + y^4$.

2. Find two factors of $c^6 - d^6$.

OPERATION.

$$(c^6 - d^6) \div (c + d) = c^5 - c^4 d + c^3 d^2 - c^2 d^3 + c d^4 - d^5$$

By II. the difference of the same even powers of two quantities is divisible by the sum of the quantities; therefore $c + d$ must be a factor of $c^6 - d^6$; and dividing $c^6 - d^6$ by $c + d$ gives the other factor $c^5 - c^4 d + c^3 d^2 - c^2 d^3 + c d^4 - d^5$.

3. Find the factors of $m^5 + n^5$.

OPERATION.

$$(m^5 + n^5) \div (m + n) = m^4 - m^3 n + m^2 n^2 - m n^3 + n^4$$

By III. the sum of the same odd powers of two quantities is divisible by the sum of the quantities; therefore $m + n$ must be a factor of $m^5 + n^5$; and dividing $m^5 + n^5$ by $m + n$ gives the other factor $m^4 - m^3 n + m^2 n^2 - m n^3 + n^4$.

4. Find the factors of $a^3 - x^3$.

Ans. $(a - x)(a^2 + a x + x^2)$.

5. Find the factors of $a^5 + x^5$.

NOTE.—In Example 2, the factors of $c^6 - d^6$ there obtained are not the only factors; for by I. $c^6 - d^6$ is divisible by $c - d$; and dividing $c^6 - d^6$ by $c - d$ gives another factor,

$$c^5 + c^4 d + c^3 d^2 + c^2 d^3 + c d^4 + d^5;$$

or by Art. 70,

$$c^6 - d^6 = (c^3 + d^3)(c^3 - d^3).$$

But $$c^5 - c^4 d + c^3 d^2 - c^2 d^3 + c d^4 - d^5,$$
$$c^5 + c^4 d + c^3 d^2 + c^2 d^3 + c d^4 + d^5,$$
$$c^3 + d^3,$$
$$c^3 - d^3,$$

are not *prime* quantities; for the first can be divided by $c - d$, and the quotient thus arising can be divided by $c^2 \pm c d + d^2$; the second can be divided by $c + d$, and the quotient thus arising will be the same as after the division of the first quantity by $c - d$, and can be divided by $c^2 \pm c d + d^2$; the third can be divided by $c + d$, and the fourth by $c - d$. Performing these divisions, by each method we shall find the prime factors of $c^6 - d^6$ to be

$c + d$, $c - d$, $c^2 + c d + d^2$, and $c^2 - c d + d^2$.

In finding the prime factors, it is better to apply first the principle of Art. 70 as far as possible.

6. Find the prime factors of $x^{10} - y^{10}$.

$$x^{10} - y^{10} = (x^5 + y^5)(x^5 - y^5).$$
$$x^5 + y^5 = (x + y)(x^4 - x^3 y + x^2 y^2 - x y^3 + y^4).$$
$$x^5 - y^5 = (x - y)(x^4 + x^3 y + x^2 y^2 + x y^3 + y^4).$$

Ans. $(x+y)(x-y)(x^4 - x^3 y + x^2 y^2 - x y^3 + y^4)(x^4 + x^3 y + x^2 y^2 + x y^3 + y^4)$.

7. Find the prime factors of $a^6 - 1$.

Ans. $(a+1)(a-1)(a^2+a+1)(a^2-a+1)$.

8. Find the prime factors of $a^4 - 2 a^2 x^2 + x^4$.

Ans. $(a + x)(a + x)(a - x)(a - x)$.

9. Find the prime factors of $x^6 + 2 x^3 y^3 + y^6$.

Ans. $(x+y)(x+y)(x^2 - x y + y^2)(x^2 - x y + y^2)$.

10. Find the prime factors of $1 - a^4$.

Ans. $(1 + a)(1 - a)(1 + a^2)$.

11. Find the prime factors of $8 - c^3$.

Ans. $(2 - c)(4 + 2 c + c^2)$.

SECTION X.

GREATEST COMMON DIVISOR.*

72. A COMMON DIVISOR of two or more quantities is any quantity that will divide each of them without remainder.

73. THE GREATEST COMMON DIVISOR of two or more quantities is the greatest quantity that will divide each of them without remainder.

74. To deduce a rule for finding the greatest common divisor of two or more quantities, we demonstrate the two following theorems:—

THEOREM I. *A common divisor of two quantities is also a common divisor of the sum or the difference of any multiples of each.*

Let A and B be two quantities, and let d be their common divisor; d is also a common divisor of $m A \pm n B$.

Suppose $A \div d = p$; i. e. $A = d p$, and $m A = d m p$,
and $B \div d = q$; i. e. $B = d q$, and $n B = d n q$;
then $m A \pm n B = d m p \pm d n q = d (m p \pm n q)$.

That is, d is contained in $m A + n B$. $m p + n q$ times, and in $m A - n B$, $m p - n q$ times; i. e. d is a common divisor of the sum or the difference of any multiples of A and B.

THEOREM II. *The greatest common divisor of two quantities is also the greatest common divisor of the less and the remainder after dividing the greater by the less.*

Let A and B be two quantities, and $A > B$; and let the process of dividing be as appears in the margin. Then, as the dividend is equal to the product of the divisor by the quotient plus the remainder,

$$\begin{array}{l} B)\ A\ (q \\ \quad\ \ \underline{q B} \\ \qquad r \end{array}$$

$$A = r + q B. \qquad (1)$$

* See Preface.

And, as the remainder is equal to the dividend minus the product of the divisor by the quotient,

$$r = A - q\,B. \qquad (2)$$

Therefore, according to the preceding theorem, from (1) any divisor of r and B must be a divisor of A; and from (2) any divisor of A and B, a divisor of r; i. e. the divisors of A and B and B and r are identical, and therefore the greatest common divisor of A and B must also be the greatest common divisor of B and r.

In the same way the greatest common divisor of B and r is the greatest common divisor of r and the remainder after dividing B by r.

Hence, to find the greatest common divisor of any two quantities,

RULE.

Divide the greater by the less, and the less by the remainder, and so continue till the remainder is zero; the last divisor is the divisor sought.

NOTE 1.—The division by each divisor should be continued until the remainder will contain it no longer.

NOTE 2.—If the greatest common divisor of more than two quantities is required, find the greatest common divisor of two of them, then of this divisor and a third, and so on; the last divisor will be the divisor sought.

NOTE 3.—The common divisor of $x\,y$ and $x\,z$ is x; x is also the common divisor of x and $x\,z$, or of $a\,x\,y$ and $x\,z$; i. e. *the common divisor of two quantities is not changed by rejecting or introducing into either any factor which contains no factor of the other.*

NOTE 4.—It is evident that the greatest common divisor of two quantities contains all the factors common to the quantities.

CASE I.

75. To find the greatest common divisor of monomials.

1. Find the greatest common divisor of $8\,a^2\,b^3\,c\,d$, $16\,a^3\,b^5\,c^2$, and $28\,a^4\,b^4\,c$.

The greatest common divisor of the coefficients found by the general rule is 4; it is evident that no higher power of a than a^2, of

b than b^3, of c than itself, will divide the quantities; and that d will not divide them; therefore, the divisor sought is $4\,a^2\,b^3\,c$. Hence,

RULE.

Annex to the greatest common divisor of the coefficients those letters which are common to all the quantities, giving to each letter the least exponent it has in any of the quantities.

2. Find the greatest common divisor of $63\,a^3\,b^7\,c^4\,d^6$, $27\,a^4\,b^8\,c^5$, and $45\,a^2\,b^9\,c^3\,d$. Ans. $9\,a^2\,b^7\,c^3$.

3. Find the greatest common divisor of $75\,x^7\,y^5\,z^3$ and $125\,a\,b\,x^3\,y^7\,z^9$.

4. Find the greatest common divisor of $99\,a\,b^2\,c^4\,d^3\,x^6\,y^5$ and $22\,a^2\,b^4\,c^3\,d^9\,x^5$. Ans. $11\,a\,b^2\,c^3\,d^3\,x^5$.

5. Find the greatest common divisor of $17\,x^4\,y^2$, $19\,x^2\,y^3$, and $212\,b\,x^7\,y^4\,z^5$.

CASE II.

76. To find the greatest common divisor of polynomials.

1. Find the greatest common divisor of $x^2 - y^2$ and $x^2 - 2\,x\,y + y^2$.

$$
\begin{array}{l}
x^2 - y^2)\;x^2 - 2\,x\,y + y^2\;(1 \\
\qquad\quad\;\; \underline{x^2 \qquad\quad\;\; - y^2} \\
\qquad\quad\;\; -2\,x\,y + 2\,y^2 \\
\text{Rejecting the factor } -2\,y \\
\qquad\qquad\qquad x - y)\;x^2 - y^2\;(x + y \\
\qquad\qquad\qquad\qquad\;\; \underline{x^2 - x\,y} \\
\qquad\qquad\qquad\qquad\qquad x\,y - y^2 \\
\qquad\qquad\qquad\qquad\qquad x\,y - y^2 \qquad \text{Ans. } x - y.
\end{array}
$$

RULE.

Arrange the terms of both quantities in the order of the powers of some letter, and then proceed according to the general rule in Art. 74.

NOTE 1. — If the leading term of the dividend is not divisible by the leading term of the divisor, it can be made so by introducing

in the dividend a factor which contains no factor of the divisor; or either quantity may be simplified by rejecting any factor which contains no factor of the other. (Art. 74, Note 3.)

NOTE 2.—Since any quantity which will divide a will divide $-a$, and *vice versa*, and any quantity divisible by a is divisible by $-a$, and *vice versa*, therefore all the signs of either divisor or dividend, or of both, may be changed from $+$ to $-$, or $-$ to $+$, without changing the common divisor.

NOTE 3.—When one of the quantities is a monomial, and the other a polynomial, either of the given rules can be applied, although generally the greatest common divisor will be at once apparent.

2. Find the greatest common divisor of $ax^7 - a^2x^6 - 8a^5x^3$ and $2cx^4 - 2acx^3 + 4a^2cx^2 - 6a^3cx - 20a^4c$.

$$ax^7 - a^2x^6 - 8a^5x^3 \;\Big|\; 2cx^4 - 2acx^3 + 4a^2cx^2 - 6a^3cx - 20a^4c$$

Dividing by ax^3 | Dividing by $2c$

$$x^4 - ax^3 - 8a^4 \;\Big)\; x^4 - ax^3 + 2a^2x^2 - 3a^3x - 10a^4 \;(1$$
$$\underline{x^4 - ax^3 \qquad\qquad\qquad - 8a^4}$$
$$2a^2x^2 - 3a^3x - 2a^4 \quad \text{1st Rem.}$$

$$2a^2x^2 - 3a^3x - 2a^4 \;\Big|\; x^4 - ax^3 - 8a^4$$

Dividing by a^2 | Multiplying by 2

$$2x^2 - 3ax - 2a^2 \;\Big)\; 2x^4 - 2ax^3 - 16a^4 \;(x^2$$
$$\underline{2x^4 - 3ax^3 - 2a^2x^2}$$
$$ax^3 + 2a^2x^2 - 16a^4$$

Multiplying by 2

$$2ax^3 + 4a^2x^2 - 32a^4 \;(ax$$
$$\underline{2ax^3 - 3a^2x^2 - 2a^3x}$$
$$7a^2x^2 + 2a^3x - 32a^4$$

Multiplying by 2

$$14a^2x^2 + 4a^3x - 64a^4 \;(7a^2$$
$$\underline{14a^2x^2 - 21a^3x - 14a^4}$$
$$25a^3x - 50a^4 \quad \text{2d Rem.}$$

$$25a^3x - 50a^4$$

Dividing by $25a^3$

$$x - 2a \;)\; 2x^2 - 3ax - 2a^2 \;(2x + a$$
$$\underline{2x^2 - 4ax}$$
$$ax - 2a^2$$
$$\underline{ax - 2a^2}$$

Ans. $x - 2a$.

3. Find the greatest common divisor of $a^4 - x^4$ and $a^3 + a^2 x - a x^2 - x^3$. Ans. $a^2 - x^2$.

4. Find the greatest common divisor of $a^4 - x^4$ and $a^5 - a^3 x^2$. Ans. $a^2 - x^2$.

5. Find the greatest common divisor of $2 a x^2 - a^2 x - a^3$ and $2 x^2 + 3 a x + a^2$.

6. Find the greatest common divisor of $6 a x - 8 a$ and $6 a x^3 + a x^2 - 12 a x$. Ans. $3 a x - 4 a$.

7. Find the greatest common divisor of $x^4 - y^4$ and $x^3 + y^3$.

8. Find the greatest common divisor of $3 x^3 - 24 x - 9$ and $2 x^3 - 16 x - 6$.

9. Find the greatest common divisor of $x^3 - y^3$ and $x^2 - y^2$. Ans. $x - y$.

10. Find the greatest common divisor of $10 x^3 - 20 x^2 y + 30 y^3$ and $x^3 + 2 x^2 y + 2 x y^2 + y^3$. Ans. $x + y$.

11. Find the greatest common divisor of $a^4 + a^3 + a^2 + a - 4$ and $a^4 + 2 a^3 + 3 a^2 + 4 a - 10$.

Ans. $a - 1$.

12. Find the greatest common divisor of $7 a x^4 + 21 a x^3 + 14 a$ and $x^5 + x^4 + x^3 - x$. Ans. $x + 1$.

13. Find the greatest common divisor of $27 a^3 y^4 - 8 a^6 y$ and $3 y^7 - 2 a y^6 + 3 a^2 y^5 - 2 a^3 y^4$.

Ans. $3 y^2 - 2 a y$.

14. Find the greatest common divisor of $a^3 + a - 10$ and $a^4 - 16$. Ans. $a - 2$.

NOTE 5. — The greatest common divisor of polynomials can also be found by factoring the polynomials, and finding the product of the factors common to the polynomials, taking each factor the least number of times it occurs in any of the quantities. (Art. 74, Note 4.)

15. Find the greatest common divisor of $3 a x^2 - 4 a x + 3 a x y - 4 a y$ and $a^3 x - x + a^3 y - y$.

$$3 a x^2 - 4 a x + 3 a x y - 4 a y = a(x + y)(3 x - 4)$$
$$a^3 x - x + a^3 y - y = (x + y)(a - 1)(a^2 + a + 1)$$

Ans. $x + y$.

SECTION XI.

LEAST COMMON MULTIPLE.

77. A MULTIPLE of any quantity is a quantity that can be divided by it without remainder.

78. A COMMON MULTIPLE of two or more quantities is any quantity that can be divided by each of them without remainder.

79. The LEAST COMMON MULTIPLE of two or more quantities is the least quantity that can be divided by each of them without remainder.

80. It is evident that a multiple of any quantity must contain the factors of that quantity; and, *vice versa,* any quantity that contains the factors of another quantity is a multiple of it: and a common multiple of two or more quantities must contain the factors of these quantities; and the *least* common multiple of two or more quantities must contain *only* the factors of these quantities.

CASE I.

To find the least common multiple of monomials.

1. Find the least common multiple of $6\,a^2\,b^2\,c$, $8\,a^3\,b^5\,c^2\,d$, and $12\,a^4\,b\,c\,x$.

The least common multiple of the coefficients, found by inspection or the rule in Arithmetic, is 24; it is evident that no quantity which contains a power of a less than a^4, of b less than b^5, of c less than c^2, and which does not contain d and x, can be divided by each of these quantities; therefore the multiple sought is $24\,a^4\,b^5\,c^2\,d\,x$.

Hence, in the case of monomials,

RULE.

Annex to the least common multiple of the coefficients all the letters which appear in the several quantities, giving to each letter the greatest exponent it has in any of the quantities.

2. Find the least common multiple of $3\,a^4\,b^2\,c^6$, $6\,a^7\,b^4\,c\,d^2$, and $10\,a\,b\,c\,x^5$. Ans. $30\,a^7\,b^4\,c^6\,d^2\,x^5$.

3. Find the least common multiple of $16\,a\,b\,x$, $80\,a\,b^4\,x^2$, and $35\,a^7\,b\,x^4$. Ans. $560\,a^7\,b^4\,x^4$.

4. Find the least common multiple of $9\,a^3\,b^5$, $15\,a^4\,b\,x^6$, and $18\,a\,x\,y^2$. Ans. $90\,a^4\,b^5\,x^6\,y^2$.

5. Find the least common multiple of $18\,a^3\,b\,c^4\,x$, $24\,a\,b^3\,c\,x^2\,y$, and $30\,a^2\,b^2\,x\,z$.

6. Find the least common multiple of $100\,x\,y\,z$, $45\,a\,b\,c$, and $25\,m\,n$.

7. Find the least common multiple of $10\,a^2\,b\,y^2$, $13\,a^4\,b^2\,c$, and $17\,a^2\,b^3\,c^2$.

8. Find the least common multiple of $14\,a^3\,b^2\,c^4$, $20\,a^7\,b\,c^4$, $25\,a^3\,b\,c^3$, and $28\,a\,b\,c\,d$.

CASE II.

81. To find the least common multiple of *any* two quantities.

Since the greatest common divisor of two quantities contains all the factors common to these quantities (Art. 74, Note 4); and since the least common multiple of two quantities must contain *only* the factors of these quantities (Art. 80); if the product of two quantities is divided by their greatest common divisor, the quotient will be their least common multiple.

Hence, to find the least common multiple of any two quantities,

RULE.

Divide one of the quantities by their greatest common divisor, and multiply this quotient by the other quantity, and the product will be their least common multiple.

NOTE 1. — If the least common multiple of more than two quantities is required, find the least common multiple of two of them, then of this common multiple and a third, and so on; the last common multiple will be the multiple sought.

NOTE 2. — In case the least common multiple of several monomials and polynomials is required, it may be better to find the least common multiple of the monomials by the Rule in Case I., and of the polynomials by the Rule in Case II., and then the least common multiple of these two multiples by the latter Rule.

1. Find the least common multiple of $x^2 - y^2$ and $x^2 - 2xy + y^2$.

OPERATION.

$$x - y)\ \underline{x^2 - 2xy + y^2}$$
$$x - y$$
$$(x^2 - y^2)(x - y), \text{ Ans.}$$

Their greatest common divisor is $x - y$, with which we divide one of the quantities; and multiplying the other quantity by this quotient, we have the least common multiple $(x^2 - y^2)(x - y)$.

2. Find the least common multiple of $2a^2x^2$, $4x^2y$, $a^4 - x^4$, and $a^5 - a^3x^2$.

The least common multiple of the monomials is $4a^2x^2y$; and the least common multiple of the polynomials is $a^3(a^4 - x^4)$.

The greatest common divisor of these two multiples is a^2; and dividing one of these multiples by a^2, and multiplying the quotient by the other, we have $4a^3x^2y(a^4 - x^4)$ as the least common multiple.

3. Find the least common multiple of $3a^2b^3$, $6a^2by$, $a^3 - 8$, and $a^2 - 4a + 4$.

Ans. $6a^2b^3y(a^3 - 8)(a - 2)$.

4. Find the least common multiple of $3x^3 - 24x - 9$ and $2x^3 - 16x - 6$.

(See 8th Example, Art. 76.)

5. Find the least common multiple of $a^4 - x^4$ and $a^5 - x^5$.

6. Find the least common multiple of $x^4 - 1$, $x^2 + 2x + 1$, and $(x - 1)^2$. Ans. $x^6 - x^4 - x^2 + 1$.

7. Find the least common multiple of $x^4 - y^4$ and $x^3 + y^3$.

8. Find the least common multiple of $a^3 + a - 10$ and $a^4 - 16$.

NOTE 3.—The least common multiple of any quantities can also be found by factoring the quantities, and finding the product of all the factors of the quantities, taking each factor the greatest number of times it occurs in any of the quantities. (Art 80.)

9. Find the least common multiple of $x^2 - 2xy + y^2$, $x^4 - y^4$, and $(x + y)^2$.

$$x^2 - 2xy + y^2 = (x - y)(x - y)$$
$$x^4 - y^4 = (x^2 + y^2)(x + y)(x - y)$$
$$(x + y)^2 = (x + y)(x + y)$$

Hence L. C. M $= (x - y)(x - y)(x^2 + y^2)(x + y)(x + y)$
$$= x^6 - x^4y^2 - x^2y^4 + y^6.$$

10 Find the least common multiple of $3ax^2 - 4ax + 3axy - 4ay$ and $a^3x - x + a^3y - y$.

(See 15th Example, Art. 76.)

Ans. $a(x + y)(3x - 4)(a^2 + a + 1)(a - 1) = 3a^4x^2 - 4a^4x + 3a^4xy - 4a^4y - 3ax^2 + 4ax - 3axy + 4ay.$

SECTION XII.

FRACTIONS.

82. When division is expressed by writing the dividend over the divisor with a line between, the expression is called a Fraction. As a fraction, the dividend is called the numerator, and the divisor the denominator.

Hence, *the value of a fraction is the quotient arising from dividing the numerator by the denominator.*

Thus, $\frac{xy}{y}$ is a fraction whose numerator is xy and denominator y, and whose value is x.

83. The principles upon which the operations in fractions are carried on are included in the following

THEOREM.

Any multiplication or division of the numerator causes a like change in the value of the fraction, and any multiplication or division of the denominator causes an opposite change in the value of the fraction.

Let $\frac{xy^2}{y}$ be any fraction; its value $= xy$.

1st. Changing the numerator.

Multiplying the numerator by y,

$$\frac{xy^3}{y} = xy^2,$$

which is y times the value of the given fraction.

Dividing the numerator by y,

$$\frac{xy}{y} = x,$$

which is $\frac{1}{y}$ of the value of the given fraction.

2d. Changing the denominator.

Multiplying the denominator by y,

$$\frac{x\,y^2}{y^2} = x,$$

which is $\frac{1}{y}$ of the value of the given fraction.

Dividing the denominator by y,

$$\frac{x\,y^2}{1} = x\,y^2,$$

which is y times the value of the given fraction.

COROLLARY.—*Multiplying or dividing both numerator and denominator by the same quantity does not change the value of the fraction.*

For if any quantity is both multiplied and divided by the same quantity its value is not changed.

$$\text{Thus, } \frac{x\,y}{y} = \frac{c\,x\,y}{c\,y} = \frac{x}{1} = x.$$

84. Every fraction has three signs: one for the numerator, one for the denominator, and one for the fraction as a whole.

$$\text{Thus, } +\frac{-a}{-b}.$$

If an even number of these signs is changed from + to —, or — to +, the value of the fraction is not changed; but if an odd number is changed, the value of the fraction is changed from + to —, or — to +.

Thus, changing an even number,

$$+\frac{+x\,y}{+y} = -\frac{-x\,y}{+y} = -\frac{+x\,y}{-y} = +\frac{-x\,y}{-y} = +x;$$

but, taking

$$+\frac{+x\,y}{+y} = +x,$$

and changing an odd number,

$$-\frac{+xy}{+y}=+\frac{-xy}{+y}=+\frac{+xy}{-y}=-\frac{-xy}{-y}=-x.$$

The various operations in fractions are presented under the following cases.

CASE I.

85. To reduce a fraction to its lowest terms.

NOTE. — A fraction is in its lowest terms when its terms are mutually prime.

1. Reduce $\frac{16\,a^2xy}{24\,a^2xy^2}$ to its lowest terms.

OPERATION.

$$\frac{16\,a^2xy}{24\,a^2xy^2}=\frac{4\,xy}{6\,xy^2}=\frac{2}{3\,y}$$

Since dividing both terms of a fraction by the same quantity does not change its value (Art. 83, Cor.), we divide both terms by any factor common to them, as $4\,a^2$; and both terms of the resulting fraction by any factor common to them, as $2\,xy$; or we can divide both terms of the given fraction by their greatest common divisor $8\,a^2xy$; the resulting fraction $\frac{2}{3\,y}$ is the fraction sought. Hence,

RULE.

Divide both terms of the fraction by any factor common to them; then divide these quotients by any factor common to them; and so proceed till the terms are mutually prime. Or,

Divide both terms by their greatest common divisor.

2. Reduce $\frac{axy}{x^2y^2}$ to its lowest terms. Ans. $\frac{a}{xy}$.

3. Reduce $\frac{272\,a^2x^2y^2}{408\,a^4xy^3}$ to its lowest terms. Ans. $\frac{2\,x}{3\,a^2y}$.

4. Reduce $\frac{24\,xyz}{12\,axy}$ to its lowest terms. Ans. $\frac{2z}{a}$.

5. Reduce $\frac{34\,a^2bxy}{51\,a^2bxy}$ to its lowest terms.

6. Reduce $\frac{108\,x^2\,y^2\,z^2}{120\,a\,b\,x^4\,y^5}$ to its lowest terms.

7. Reduce $\frac{x^2 - y^2}{x^2 + 2\,x\,y + y^2}$ to its lowest terms.

Ans. $\frac{x - y}{x + y}$.

8. Reduce $\frac{a^2 - b^2}{a^2 - 2\,a\,b + b^2}$ to its lowest terms.

9. Reduce $\frac{a\,x^7 - a^2\,x^6}{2\,c\,x^4 - 2\,a\,c\,x^3 + 4\,a^2\,c\,x^2 - 4\,a^3\,c\,x}$ to its lowest terms.

Ans. $\frac{a\,x^5}{c\,(2\,x^2 + 4\,a^2)}$.

10. Reduce $\frac{a^3 + a^2\,x - a\,x^2 - x^3}{a^4 - x^4}$ to its lowest terms.

CASE II.

86. To reduce fractions to equivalent fractions having a common denominator.

1. Reduce $\frac{a}{b\,y}$ and $\frac{c}{b\,x}$ to equivalent fractions having a common denominator.

OPERATION.

$$\frac{a}{b\,y} = \frac{a\,b\,x}{b^2\,x\,y}$$

$$\frac{c}{b\,x} = \frac{b\,c\,y}{b^2\,x\,y}$$

We multiply the numerator and denominator of each fraction by the denominator of the other (Art. 83, Cor.). This must reduce them to equivalent fractions having a common denominator, as the new denominator of each fraction is the product of the same factors.

OR,

$$\frac{a}{b\,y} = \frac{a\,x}{b\,x\,y}$$

$$\frac{c}{b\,x} = \frac{c\,y}{b\,x\,y}$$

In the second operation we find the least common multiple, $b\,x\,y$, of the denominators $b\,y$ and $b\,x$; as each denominator is contained in this multiple, each fraction can be reduced to a fraction with this multiple as a denominator, by multiplying its numerator and denominator by the quotient arising from dividing this multiple by its denominator. Hence,

RULE.

Multiply all the denominators together for a common denominator, and multiply each numerator into the continued product of all the denominators, except its own, for new numerators. Or,

Find the least common multiple of the denominators for the least common denominator. For new numerators, multiply each numerator by the quotient arising from dividing this multiple by its denominator.

2. Reduce $\frac{m}{xy}$, $\frac{n}{ab}$, and $\frac{x}{aby}$ to equivalent fractions having the least common denominator.

Ans. $\frac{abm}{abxy}$, $\frac{nxy}{abxy}$, and $\frac{x^2}{abxy}$.

3. Reduce $\frac{8a}{15b}$, $\frac{3xy}{10bc}$, and $\frac{2x}{25acd}$ to equivalent fractions having the least common denominator.

Ans. $\frac{80a^2cd}{150abcd}$, $\frac{45adxy}{150abcd}$, and $\frac{12bx}{150abcd}$.

4. Reduce $\frac{ax}{m}$, $\frac{abc}{nxy}$, and $\frac{8}{5d}$ to equivalent fractions having the least common denominator.

5. Reduce $\frac{a-b}{a+b}$ and $\frac{am}{a-b}$ to equivalent fractions having the least common denominator.

Ans. $\frac{a^2-2ab+b^2}{a^2-b^2}$ and $\frac{a^2m+abm}{a^2-b^2}$.

6. Reduce $\frac{x+1}{x-4}$ and $\frac{a+m}{x-1}$ to equivalent fractions having the least common denominator.

7. Reduce $\frac{ab}{x^2-y^2}$, $\frac{c}{x+y}$, and $\frac{d}{x-y}$ to equivalent fractions having the least common denominator.

CASE III.

87. To add fractions.

1. Find the sum of $\frac{b}{x}$ and $\frac{c}{x}$.

OPERATION.

$$\frac{b}{x}+\frac{c}{x}=\frac{b+c}{x}$$

If anything is divided into equal parts, a number of these parts represented by b, added to a number represented by c, gives $b+c$ of these parts. In the example given, a unit is divided into x equal parts, and it is required to find the sum of b and c of these parts; i. e. $\frac{b}{x}+\frac{c}{x}=\frac{b+c}{x}$.

It is evident, therefore, that fractions that have a common denominator can be added by adding their numerators. But fractions that do not have a common denominator can be reduced to equivalent fractions having a common denominator. Hence,

RULE.

Reduce the fractions, if necessary, to equivalent fractions having a common denominator; then write the sum of the numerators over the common denominator.

2. Add $\frac{m}{n}$, $\frac{x}{y}$, and $\frac{a}{b}$. Ans. $\frac{bmy+bnx+any}{bny}$.

3. Add $\frac{2a}{7}$, $\frac{3x}{4b}$, and $\frac{c}{2d}$.

4. Add $\frac{3ab}{4xy}$, $\frac{2xy}{5ab}$, and $\frac{7m}{8abxy}$. Ans. $\frac{30a^2b^2+16x^2y^2+35m}{40abxy}$.

5. Add $\frac{8x^2y}{3a^2}$, $\frac{5a^2b}{9cd}$, and $\frac{4x^2z}{27ac}$.

6. Add $\frac{1}{1+a}$ and $\frac{1}{1-a}$. Ans. $\frac{2}{1-a^2}$.

7. Add $\frac{1+a}{1-a}$ and $\frac{1-a}{1+a}$. Ans. $\frac{2+2a^2}{1-a^2}$.

8. Add $\frac{1+x^2}{7}$ and $\frac{7}{1+x^2}$.

9. Add $\frac{x^2+y^2}{7(x+y)}$ and $\frac{8x^2}{x-y}$.

10. Add $\frac{x-y}{xy}$, $\frac{y-z}{yz}$, and $\frac{z-x+xz}{xz}$. Ans. 1.

11. Add $\frac{1}{x^2-y^2}$ and $\frac{2xy}{x^4-y^4}$.

Ans. $\frac{x+y}{x^3-x^2y+xy^2-y^3}$.

12. Add mx and $\frac{7x}{18a}$.

NOTE.—Consider $mx = \frac{mx}{1}$, and then proceed as before.

Ans. $\frac{18amx+7x}{18a}$.

13. Add $x+y$ and $\frac{7x}{a+b}$.

14. Add $x^2+2xy+y^2$ and $\frac{1}{x-y}$.

Ans. $\frac{x^3+x^2y-xy^2-y^3+1}{x-y}$.

CASE IV.

88. To subtract one fraction from another.

1. Subtract $\frac{c}{x}$ from $\frac{b}{x}$.

OPERATION.

$$\frac{b}{x}-\frac{c}{x}=\frac{b-c}{x}$$

If anything is divided into x equal parts, a number of these parts represented by c, subtracted from a number represented by b, leaves $b-c$ of these parts; i. e. $\frac{b}{x}-\frac{c}{x}=\frac{b-c}{x}$. Hence,

RULE.

Reduce the fractions, if necessary, to equivalent fractions having a common denominator; then subtract the numerator of the subtrahend from that of the minuend, and write the result over the common denominator.

2. Subtract $\frac{7x}{4}$ from $\frac{ab}{8c}$. Ans. $\frac{ab - 14cx}{8c}$.

3. Subtract $\frac{3a}{7x}$ from $\frac{7ab}{6ax}$.

4. Subtract $\frac{14ab}{19x^2}$ from $\frac{15c}{19a}$.

5. Subtract $\frac{29ac}{14x^2}$ from $\frac{39x}{8xy}$. Ans. $\frac{273x^2 - 116acy}{56x^2y}$.

6. Subtract $\frac{1}{1-a}$ from $\frac{1}{1+a}$. Ans. $\frac{2a}{a^2-1}$.

7. Subtract $\frac{x+1}{x-1}$ from $\frac{x-1}{x+1}$.

8. Subtract $\frac{ab+bc}{a^2x - b^2x}$ from $\frac{ab - bc}{a^2x^2 - b^2x^2}$.

9. Subtract $\frac{1}{a-b}$ from $\frac{1}{a+b}$. Ans. $\frac{2b}{b^2-a^2}$.

10. Subtract $\frac{4x^2}{x^4-1}$ from $\frac{x^2+1}{x^2-1}$. Ans. $\frac{x^2-1}{x^2+1}$.

11. Subtract 16 from $\frac{x^2-7}{1+x}$.

NOTE. — Consider $16 = \frac{16}{1}$, and then proceed as before.

Ans. $\frac{x^2 - 16x - 23}{1+x}$.

12. Subtract $\frac{x^2-3}{a-b}$ from xy.

13. Subtract $x+b$ from $\frac{bx+4b}{b+4}$. Ans. $-\frac{b^2+4x}{b+4}$.

CASE V.

89. To reduce a mixed quantity to an improper fraction.

1. Reduce $x + \frac{a}{8}$ to an improper fraction.

OPERATION.

$$x + \frac{a}{8} = \frac{8x}{8} + \frac{a}{8} = \frac{8x+a}{8}$$

As eight eighths make a unit, there will be in x units eight times x eighths; i. e. $x = \frac{8x}{8}$; and $\frac{8x}{8} + \frac{a}{8} = \frac{8x+a}{8}$. Hence,

RULE.

Multiply the integral part by the denominator of the fraction; to the product add the numerator if the sign of the fraction is plus, and subtract it if the sign is minus, and under the result write the denominator.

NOTE.—By a change of the language, Examples 12–14 in Art. 87, and 11–13 in Art. 88, become examples under this case. Thus, Example 12, Art. 87, might be expressed as follows: Reduce $m x + \frac{7x}{18a}$ to an improper fraction.

2. Reduce $x^2 + 4 - \frac{7}{y}$ to an improper fraction.

Ans. $\frac{x^2 y + 4y - 7}{y}$.

3. Reduce $25a - 25x + \frac{a+x}{2}$ to an improper fraction.

4. Reduce $a - 1 + \frac{1-a}{1+a}$ to an improper fraction.

Ans. $\frac{a^2 - a}{a+1}$.

5. Reduce $y + \frac{2xy + x^2}{y}$ to an improper fraction.

6. Reduce $\frac{a^2 + b^2}{b} - (a + b)$ to an improper fraction.

Ans. $\frac{a^2 - ab}{b}$.

7. Reduce $x - 1 - \frac{x^2+1}{x+1}$ to an improper fraction.

NOTE.—It must be remembered that the sign before the dividing line belongs to the fraction as a whole.

$$x - 1 - \frac{x^2+1}{x+1} = \frac{x^2 - 1 - x^2 - 1}{x+1} = \frac{-2}{x+1}, \text{ or } -\frac{2}{x+1}, \quad \text{Ans.}$$

8. Reduce $x + 1 - \frac{x^2+1}{x-1}$ to an improper fraction.

Ans. $\frac{2}{1-x}$.

9. Reduce $x^2 - 2ax + a^2 - \frac{2x^2 - a^2}{x - a}$ to an improper fraction.

NOTE.— According to the same principle an integral quantity can be reduced to a fraction having any given denominator, by *multiplying the quantity by the proposed denominator, and under the product writing the denominator.*

10. Reduce $x + 1$ to a fraction whose denominator is $x - 1$.
Ans. $\frac{x^2 - 1}{x - 1}$.

11. Reduce $x - 1$ to a fraction whose denominator is $a - b$.

12. Reduce $4ax$ to a fraction whose denominator is $a^2 - z$.

CASE VI.

90. To reduce an improper fraction to an integral or mixed quantity.

1. Reduce $\frac{x^2 - 4ax + 5a^2}{x - 2a}$ to an integral or mixed quantity.

OPERATION.

$$\begin{array}{r|l} x - 2a) & x^2 - 4ax + 5a^2 \left(x - 2a + \frac{a^2}{x - 2a}\right. \\ & \underline{x^2 - 2ax \qquad\qquad} \\ & -2ax + 5a^2 \\ & \underline{-2ax + 4a^2} \\ & a^2 \end{array}$$

As the value of a fraction is the quotient arising from dividing the numerator by the denominator (Art. 82), we perform the indicated division. Hence,

RULE.

Divide the numerator by the denominator; if there is any remainder, place it over the divisor, and annex the fraction so formed with its proper sign to the quotient.

2. Reduce $\frac{a x - 4 b x}{x}$ to an integral or mixed quantity.

Ans. $a - 4 b$.

3. Reduce $\frac{4 a b - 3 a b x + y}{a b}$ to an integral or mixed quantity.

4. Reduce $\frac{4 x^2 - 4 y^2}{2 x - 2 y}$ to an integral or mixed quantity.

5. Reduce $\frac{24 y^3 - 12 a y^2 + 7}{6 y^2}$ to an integral or mixed quantity.

Ans. $4 y - 2 a + \frac{7}{6 y^2}$.

6. Reduce $\frac{x^2 + a x + a^2}{x}$ to an integral or mixed quantity.

7. Reduce $\frac{8 a x - 10 b x - 5 c x}{2 x}$ to an integral or mixed quantity.

8. Reduce $\frac{4 a^2 - 8 a b + 4 b^2 - 2 a}{2 a - 2 b}$ to an integral or mixed quantity.

Ans. $2 a - 2 b - \frac{a}{a - b}$.

9. Reduce $\frac{x^7 - y^7}{x - y}$ to an integral or mixed quantity.

10. Reduce $\frac{x^6 - y^6}{x - y}$ to an integral or mixed quantity

CASE VII.

91. To multiply a fraction by an integral quantity.

1. Multiply $\frac{x + y}{a b}$ by c.

OPERATION.

$$\frac{x + y}{a b} \times c = \frac{c x + c y}{a b}$$

According to the theorem in Art. 83, multiplying the numerator by c multiplies the value of the fraction c times.

2. Multiply $\frac{x+y}{ab}$ by a.

OPERATION.

$$\frac{x+y}{ab} \times a = \frac{x+y}{b}$$

According to the theorem in Art. 83, dividing the denominator by a multiplies the value of the fraction a times. Hence,

RULE.

Divide the denominator by the integral quantity when it can be done without remainder; otherwise, multiply the numerator by the integral quantity.

3. Multiply $\frac{3ax+4xy}{m^2-n^2}$ by $m+n$.
Ans. $\frac{3ax+4xy}{m-n}$.

4. Multiply $\frac{17x^3-4}{3ax+b}$ by ab.

5. Multiply $\frac{x^4-a}{3b+3c}$ by $3y$.

NOTE. — Any factor common to the denominator and multiplier may be cancelled from both before multiplying.

$$\frac{x^4-a}{3b+3c} \times 3y = \frac{x^4-a}{b+c} \times y = \frac{x^4y-ay}{b+c}, \text{ Ans.}$$

6. Multiply $\frac{x^4-1}{ab+ax}$ by $7a$.

7. Multiply $\frac{a+x}{7(x^2+y^2)}$ by $14(x^4-y^4)$.
Ans. $2(x^2-y^2)(a+x)$.

8. Multiply $\frac{x+y}{x-y}$ by $x-y$.

NOTE. — When a fraction is multiplied by a quantity equal to its denominator, the product is the numerator.

$$\frac{x+y}{x-y} \times (x-y) = \frac{x+y}{1} = x+y, \text{ Ans.}$$

9. Multiply $\frac{x^2 + 2xy + y^2}{(x - a)^2}$ by $(x - a)^2$.

10. Multiply $\frac{a + b}{x - y}$ by $x^2 - 2xy + y^2$.

Ans. $(a + b)(x - y)$.

CASE VIII.

92. To multiply an integral quantity by a fraction.

1. Multiply $x^2 + 2xy + y^2$ by $\frac{4}{x + y}$.

OPERATION.

$$(x^2 + 2xy + y^2) \times 4 = 4(x^2 + 2xy + y^2)$$
$$4(x^2 + 2xy + y^2) \div (x + y) = 4(x + y)$$

We first multiply the multiplicand by the numerator 4; but the multiplier is $4 \div (x + y)$; and therefore this product is $x + y$ times too great, and this product divided by $x + y$ must be the product sought.

It is evident that the result would be the same if the division were performed first, and the multiplication afterward. Hence,

RULE.

Divide the integral quantity by the denominator when it can be done without remainder, and multiply the quotient by the numerator. Otherwise, multiply the integral quantity by the numerator, and divide the product by the denominator.

2. Multiply $a^3 - 3a^2b + 3ab^2 - b^3$ by $\frac{7x}{a^2 - 2ab + b^2}$.

Ans. $7x(a - b)$.

3. Multiply $a^4 - x^4$ by $\frac{3}{a^2 + x^2}$.

4. Multiply $7a^3 - 4xy$ by $\frac{3}{3y^2 - 4}$.

Ans. $\frac{21a^3 - 12xy}{3y^2 - 4}$.

5. Multiply $17(x^2 - y^2)$ by $\frac{x + y}{x - y}$.

NOTE. — Since the product is the same, whichever quantity is considered as the multiplier, by considering the integral quantity as the multiplier, Case VIII. becomes the same as Case VII.

CASE IX.

93. To divide a fraction by an integral quantity.

1. Divide $\frac{a}{b}$ by a.

OPERATION.

$$\frac{a}{b} \div a = \frac{1}{b}$$

According to the theorem in Art. 83, dividing the numerator by a decreases the value of the fraction a times.

2. Divide $\frac{a}{b}$ by c.

OPERATION.

$$\frac{a}{b} \div c = \frac{a}{b\,c}$$

According to the theorem in Art. 83, multiplying the denominator by c decreases the value of the fraction c times. Hence,

RULE.

Divide the numerator by the integral quantity when it will divide it without remainder; otherwise, multiply the denominator by the integral quantity.

3. Divide $\frac{a\,x}{4\,b\,c}$ by a. Ans. $\frac{x}{4\,b\,c}$.

4. Divide $\frac{7\,x^2}{y^4}$ by $14\,y^2$. Ans. $\frac{x^2}{2\,y^6}$.

5. Divide $\frac{6\,a\,b\,c}{25\,x\,y}$ by $6\,a\,b\,c$.

6. Divide $\frac{27\,a\,x}{32\,b\,y^4}$ by $9\,a\,b\,y^2$. Ans. $\frac{3\,x}{32\,b^2\,y^6}$.

7. Divide $\frac{4\,(a+x)}{13\,(x+y)}$ by $2\,(a+x)\,(x+y)$.

Ans. $\frac{2}{13\,(x+y)^2}$.

CASE X.

94. To divide an integral quantity by a fraction.

1. Divide x by $\frac{a}{b}$.

OPERATION.

$$x \div a = \frac{x}{a}$$

$$\frac{x}{a} \times b = \frac{b\,x}{a}$$

$x \div a = \frac{x}{a}$; but the divisor is not a, but $a \div b$. Dividing by a, therefore, is dividing by a divisor b times too great, and the quotient will be b times too small; therefore the quotient sought is $\frac{x}{a} \times b = \frac{b\,x}{a}$. Hence,

RULE.

Divide the integral quantity by the numerator, and multiply the quotient by the denominator.

2. Divide $4\,a\,x$ by $\frac{3\,x}{4}$. Ans. $\frac{16\,a}{3}$.

3. Divide $7\,x^2$ by $\frac{x^4}{a\,b\,c}$. Ans. $\frac{7\,a\,b\,c}{x^2}$.

4. Divide $a + b$ by $\frac{x}{y}$. Ans. $\frac{a\,y + b\,y}{x}$.

5. Divide $a^2 + 2\,a\,x + x^2$ by $\frac{a + x}{4}$.

6. Divide $x^4 - b\,x^2$ by $\frac{x^2}{2}$.

7. Divide $2\,x^7 + 3\,y^4$ by $\frac{3\,x + y}{4}$.

8. Divide 1 by $\frac{x}{y}$. Ans. $\frac{y}{x}$.

NOTE. — Hence, *the reciprocal of a fraction is the fraction inverted.*

CASE XI.

95. To multiply a fraction by a fraction.

1. Multiply $\frac{a}{b}$ by $\frac{x}{y}$.

OPERATION.

$$\frac{a}{b} \times x = \frac{ax}{b}$$

$$\frac{ax}{b} \div y = \frac{ax}{by}$$

We first multiply $\frac{a}{b}$ by x; but the multiplier is not x, but $x \div y$; therefore the product is y times too great; and $\frac{ax}{b} \div y = \frac{ax}{by}$ (Art. 93) must be the product sought. Hence,

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

NOTE 1.—Common factors in the numerators and denominators may be cancelled before multiplication.

NOTE 2.—Cases VII. and VIII. can be included in this by writing the integral quantity as the numerator of a fraction, with a unit as the denominator.

NOTE 3.—Mixed quantities may be reduced to improper fractions before multiplying.

2. Multiply $\frac{ax}{bc}$ by $\frac{mn}{dy}$. Ans. $\frac{amnx}{bcdy}$.

3. Multiply $\frac{x^2 y}{4ab}$ by $\frac{x^3 y^2}{2a^2 c}$.

4. Multiply $\frac{m^2 x}{abc}$ by $\frac{ac}{mx}$.

5. Multiply $\frac{a+b}{b-c}$ by $\frac{b-c}{4}$. Ans. $\frac{a+b}{4}$.

6. Multiply $\frac{x-y}{4xyz}$ by $\frac{2xy}{x-y}$.

7. Multiply $\frac{a+b}{ab+bc+bd}$ by $\frac{b^2}{a+b}$.

8. Multiply $\frac{x^2+y^2}{y-1}$ by $\frac{2y^2}{y+1}$.

9. Multiply $\frac{2a^2-4a^3}{21y}$ by $\frac{7y}{2a^3-8a^4}$. Ans. $\frac{1}{3+6a}$.

10. Multiply $\frac{13a^2x^2 - 14ay}{7x^2y^2 - 4}$ by $\frac{7x^2y^2 - 4}{13a^2x^2 - 14ay}$.

11. Multiply $\frac{4x^2}{7x + 14y}$ by $\frac{7}{4x^2}$. Ans. $\frac{1}{x + 2y}$.

12. Multiply $\frac{x^2 - a^2}{dx}$ by $\frac{x^2 + a^2}{d + x}$.

13. Multiply $y + \frac{ay}{a - y}$ by $y - \frac{ay}{a + y}$.

Ans. $\frac{2ay^3 - y^4}{a^2 - y^2}$.

14. Multiply together $\frac{x - y}{a}$, $\frac{a}{x + y}$, and $\frac{x + y}{x - y}$.

Ans. 1.

15. Multiply together $\frac{a + b}{4x}$, $\frac{2x}{a^2 - b^2}$, and $\frac{2x}{3y}$.

16. Multiply together $a + \frac{b}{y}$, $b + \frac{a}{y}$, and $y - \frac{a}{b}$.

Ans. $aby + b^2 - \frac{a^3}{by} - \frac{a^2}{y^2}$.

CASE XII.

96. To divide a fraction by a fraction.

1. Divide $\frac{x}{y}$ by $\frac{a}{b}$.

OPERATION.

$$\frac{x}{y} \div a = \frac{x}{ay}$$

$$\frac{x}{ay} \times b = \frac{bx}{ay}$$

$\frac{x}{y} \div a = \frac{x}{ay}$ (Art. 93); but the divisor is not a, but $\frac{a}{b}$; we have used a divisor b times too great, and therefore the quotient $\frac{x}{ay}$ is b times too small, and the quotient sought is $\frac{x}{ay} \times b = \frac{bx}{ay}$ (Art. 91). It will be noticed that the denominator of the dividend is multiplied by the numerator of the divisor, and the numerator of the dividend by the denominator of the divisor. Hence,

RULE.

Invert the divisor, and then proceed as in multiplication of a fraction by a fraction.

NOTE 1. — All cases in division of fractions can be brought under this rule, by writing integral quantities as fractions with a unit for the denominator.

NOTE 2. — After the divisor is inverted, common factors can be cancelled, as in multiplication of fractions.

NOTE 3. — Mixed quantities should be reduced to improper fractions before division.

2. Divide $\frac{b}{c}$ by $\frac{m}{n}$. Ans. $\frac{b\,n}{c\,m}$.

3. Divide $\frac{x^2}{y}$ by $\frac{x^4}{4}$. Ans. $\frac{4}{x^2\,y}$.

4. Divide $\frac{8\,a^2\,x}{17\,y}$ by $\frac{4\,a\,x^3}{34\,y^4}$.

5. Divide $\frac{2\,a^2 + 2\,c}{2\,b^3 + a}$ by $\frac{3\,a}{5\,c}$. Ans. $\frac{10\,a^2\,c + 10\,c^2}{6\,a\,b^3 + 3\,a^2}$.

6. Divide $\frac{3\,x\,y + y^2}{m^3 - n^3}$ by $\frac{y}{m - n}$.

7. Divide $\frac{x + y}{x^3 + y}$ by $\frac{x^2 + x\,y}{x - y}$. Ans. $\frac{x - y}{x^3 + x\,y}$.

8. Divide $\frac{a^3 + b^3}{a - b}$ by $\frac{a^2 - b^2}{4}$.

9. Divide $\frac{3\,x^2}{x^3 + y^3}$ by $\frac{x}{x + y}$. Ans. $\frac{3\,x}{x^2 - x\,y + y^2}$.

10. Divide $\frac{6\,x - 7}{x + 1}$ by $\frac{x - 1}{4}$.

11. Divide $\frac{x + x^2}{4\,a^2}$ by $\frac{2\,a\,c + 4\,a^2\,c}{3\,b}$.

12. Divide $\frac{3\,m^6\,n - 3\,n^7}{m^2\,n - 2\,m\,n^2 + n^3}$ by $\frac{m^2 + m\,n + n^2}{m - n}$.
Ans. $3\,(m^3 + n^3)$.

13. Divide $1 + \frac{x + y}{c}$ by $\frac{c}{x + y} - 1$.
Ans. $\frac{c\,(x + y) + (x + y)^2}{c\,(c - x - y)}$.

14. Divide $x + y - \frac{1}{d}$ by $\frac{1}{d} + x + y$.

15. Divide $\frac{1+x}{1-x}+\frac{1-x}{1+x}$ by $\frac{1-x^2}{1+x^2}$.

Ans. $\frac{2(1+x^2)^2}{(1-x^2)^2}$.

NOTE.—The division of fractions is sometimes expressed by writing the divisor under the dividend. Thus, $\frac{\frac{a}{b}}{\frac{x}{y}}$. Such an expression is called a COMPLEX FRACTION. A Complex Fraction can be reduced to a simple one by performing the division indicated.

16. Reduce $\frac{\frac{a}{x}}{\frac{7}{5}}$ to a simple fraction. Ans. $\frac{5a}{7x}$.

17. Reduce $\frac{1+\frac{a}{x}}{7-\frac{b}{c}}$ to a simple fraction.

Ans. $\frac{cx+ac}{7cx-bx}$.

18. Reduce $\frac{\frac{x+1}{x-1}}{\frac{x-1}{x+1}}$ to a simple fraction.

19. Reduce $\frac{x^2-\frac{1}{y}}{\frac{x+4}{1-x}}$ to a simple fraction.

20. Reduce $\frac{\frac{a^2-b^2}{x+y}}{a+b}$ to a simple fraction.

Ans. $\frac{a-b}{x+y}$.

21. Reduce $\frac{\frac{x^2-y^2}{x-y}}{\frac{1}{a+b}}$ to a simple fraction.

Ans. $(x+y)(a+b)$.

NOTE.—A Complex Fraction can also be reduced by multiplying its numerator and denominator by the least common multiple of the denominators of the fractional parts. Thus, if both terms of the fraction in Ex. 16 be multiplied by $5x$, or both in Ex. 17 by cx, the result will be the same as above.

SECTION XIII.

EQUATIONS

OF THE FIRST DEGREE CONTAINING BUT ONE UNKNOWN QUANTITY.

97. An EQUATION is an expression of equality between two quantities (Art. 9). That portion of the equation which precedes the sign $=$ is called *the first member*, and that which follows, *the second member*.

98. The DEGREE of an equation containing but one unknown quantity is denoted by the exponent of the highest power of the unknown quantity in the equation.

An equation of the *first degree*, or a *simple equation*, is one that contains only the *first* power of the unknown quantity. For example,

$$2x - ax = 27.$$

An equation of the *second degree*, or a *quadratic equation*, is one in which the highest power of the unknown quantity is the *second* power. For example,

$$x^2 - ax = b + c, \text{ or } ax^2 - b = 17.$$

An equation of the *third degree*, or a *cubic equation*, is one in which the highest power of the unknown quantity is the *third* power, and so on.

99. The REDUCTION OF AN EQUATION consists in finding the value of the unknown quantity, and the processes involved depend upon the Axioms given in Art. 13. The processes can be best understood by considering an equation as a pair of scales which balance as long as an equal weight remains in both sides: whenever on one side any additional weight is put in or taken out, an equal weight must be put in or

taken out on the other side, in order that the equilibrium may remain. So, *in an equation, whatever is done to one side must be done to the other*, in order that the equality may remain.

1. If anything is added to one member, an equal quantity must be added to the other.

2. If anything is subtracted from one member, an equal quantity must be subtracted from the other.

3. If one member is multiplied by any quantity, the other member must be multiplied by an equal quantity.

4. If one member is divided by any quantity, the other member must be divided by an equal quantity.

5. If one member is involved or evolved, the other must be involved or evolved to the same degree.

TRANSPOSITION.

100. TRANSPOSITION is the changing of terms from one member of an equation to the other, without destroying the equality.

The object of transposition is to bring all the unknown terms into one member and all the known into the other, so that the unknown may become known.

1. Find the value of x in the equation $x + 16 = 24$.

OPERATION.

$$x + 16 = 24$$
$$x = 24 - 16 = 8$$

Subtracting 16 from the first member leaves x; but if 16 is subtracted from the first member, it must also be subtracted from the second.

2 Find the value of x in the equation $x - b = a$.

OPERATION.

$$x - b = a$$
$$x = a + b$$

Adding b to the first member gives x; but if b is added to the first member it must also be added to the second.

3. Find the value of x in the equation $2x = x + 16$.

OPERATION.

$$2x = x + 16$$
$$2x - x = 16$$
$$x = 16$$

Subtracting x from both members, we have $2x - x = 16$, or $x = 16$.

It appears from these examples that any term which disappears from one member of an equation reappears in the other with the opposite sign. Hence,

RULE.

Any term may be transposed from one member of an equation to the other, provided its sign is changed.

4. Find the value of x in the equation $8x - 15 = 4x + 5$.

OPERATION.

$$8x - 15 = 4x + 5$$

Transposing, $8x - 4x = 5 + 15$

Uniting terms, $4x = 20$

Dividing both members by 4, $x = 5$

5. Find the value of x in $4x + 46 = 5x + 23$.

NOTE. — Reducing, we have $-x = -23$. If each member of this equation is transposed, we shall have $23 = x$; i. e. 23 equals x, or x equals 23. Dividing both members by -1 will give the same result. Hence, *the signs of all the terms of an equation may be changed without destroying the equality.*

6. Find the value of x in $17x + 17 = 19x + 13$.

Ans. $x = 2$.

7. Find the value of x in $8x - 14 = 13x - 29$.

8. Find the value of x in $5x + 25 = 10x - 25$.

Ans. $x = 10$.

9. Find the value of x in $24x - 17 = 11x + 74$.

10. Find the value of x in $37x - (4 + 7) = 41x - 23$.

CLEARING OF FRACTIONS.

131. To clear an equation of fractions.

1. Find the value of x in the equation $\frac{x}{3} - 2 = \frac{x}{6} + 1$.

OPERATION.

$$\frac{x}{3} - 2 = \frac{x}{6} + 1$$
$$2x - 12 = x + 6$$
$$x = 18$$

If the given equation is multiplied by 6, the least common multiple of 6 and 3, it will give $2x - 12 = x + 6$, an equation without a fractional term. Hence,

RULE.

Multiply each term of the equation by the least common multiple of the denominators.

NOTE 1. — In multiplying a fractional term, divide the multiplier by the denominator of the fraction and multiply the numerator by the quotient.

NOTE 2. — An equation may be cleared of fractions by multiplying it first by one denominator, and the resulting equation by another, and so on, till all the denominators disappear; but multiplying by the least common multiple is generally the more expeditious method.

NOTE 3. — Before clearing of fractions it is better to unite terms which can readily be united; for instance, the equation in Ex. 1, by transposing -2, can be written $\frac{x}{3} = \frac{x}{6} + 3$.

NOTE 4. — When the sign $-$ is before a fraction and the denominator is removed, the sign of each term that was in the numerator must be changed.

2. Given $\frac{x}{4} - \frac{x}{5} + 25 = 33 - \frac{x - 6}{2}$.

OPERATION.

Transposing 25,	$\frac{x}{4} - \frac{x}{5} = 8 - \frac{x-6}{2}$
Multiplying by 20,	$5x - 4x = 160 - 10x + 60$
Transposing and uniting,	$11x = 220$
Dividing by 11,	$x = 20$

NOTE. — The sign of the numerator of $-\frac{x}{5}$ is $+$, and must be changed to $-$ when the denominator is removed; for $-(+4x) = -4x$; and so the sign of each term of the numerator of the fraction $-\frac{x-6}{2}$ must be changed when the denominator 2 is removed; for

$$-(+10x - 60) = -10x + 60.$$

102. To reduce an equation of the first degree containing but one unknown quantity, we deduce from the preceding examples the following

RULE.

Clear the equation of fractions, if necessary.

Transpose the known terms to one member and the unknown to the other, and reduce each member to its simplest form.

Divide both members by the coefficient of the unknown quantity.

NOTE 1. — To verify an equation, we have only to substitute in the equation the value of the unknown quantity found by reducing the equation. For instance, in Ex. 2, Art. 101, by substituting 20 for x, in $\frac{x}{4} - \frac{x}{5} + 25 = 33 - \frac{x-6}{2}$, we have

$$\frac{20}{4} - \frac{20}{5} + 25 = 33 - \frac{20-6}{2},$$

$$5 - 4 + 25 = 33 - 7,$$

$$26 = 26.$$

NOTE 2. — When answers are not given, the work should be verified.

103. Since the relations between quantities in Algebra are often expressed in the form of a proportion, we introduce here the necessary definitions.

104. RATIO is the relation of one quantity to another of the same kind; or, it is the quotient which arises from dividing one quantity by another of the same kind.

Ratio is indicated by writing the two quantities after one another with two dots between, or by expressing the division in the form of a fraction. Thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$; read, a is to b, or a divided by b.

105. PROPORTION is an equality of ratios. Four quantities are proportional when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=) or by four dots (::).

Thus, $a : b = c : d$, or $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$: read, a to b equals c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

The first and fourth terms of a proportion are called the *extremes*, and the second and third the *means*.

106. In a proportion the product of the means is equal to the product of the extremes.

Let $\quad a : b = c : d$

i. e. $\quad \frac{a}{b} = \frac{c}{d}$

Clearing of fractions, $\quad ad = bc$

A proportion is an equation; and making the product of the means equal to the product of the extremes is merely clearing the equation of fractions.

EXAMPLES.

1. Reduce $\frac{4x}{5} + 10 = \frac{7x}{10} + 13$. Ans. $x = 30$.

2. Reduce $17x - 14 = 12x - 4$. Ans. $x = 2$.

3. Reduce $6x - 25 + x = 135 - 3x - 10$.

Ans. $x = 15$.

4. Reduce $3x + 5 - x = 38 - 2x$. Ans. $x = 8\frac{1}{4}$.

5. Reduce $\frac{x-4}{2} + \frac{x}{3} = 30 - \frac{x+32}{2}$. Ans. $x = 12$.

6. Reduce $x - 7\frac{1}{2} = \frac{31}{4} - \frac{3x}{8}$. Ans. $x = 11\frac{1}{11}$.

7. Reduce $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 154$. Ans. $x = 120$.

8. Reduce $\frac{x}{2} + \frac{x}{3} = 16 + \frac{x}{6}$. Ans. $x = 24$.

9. Reduce $\frac{x}{b} + a = \frac{x}{h} - \frac{x}{c} + d$.

OPERATION.

$$\frac{x}{b} + a = \frac{x}{h} - \frac{x}{c} + d$$

Multiplying by bch, $chx + abch = bcx - bhx + bcdh$

Transposing, $chx - bcx + bhx = bcdh - abch$

Factoring 1st mem., $(ch - bc + bh)x = bcdh - abch$

Dividing by coefficient of x, $x = \frac{bcdh - abch}{ch - bc + bh}$

10. Reduce $x + mx = c$. Ans. $x = \frac{c}{1+m}$.

11. Reduce $\frac{6-c}{x} - 3 = 7$. Ans. $x = \frac{6-c}{10}$.

12. Reduce $\frac{1}{a} + \frac{b}{c} = x$. Ans. $x = \frac{ab+c}{ac}$.

13. Reduce $\frac{a}{x} + \frac{b}{x} = c$. Ans. $x = \frac{a+b}{c}$.

14. Reduce $8 = \frac{14}{x-2} + 6$. Ans. $x = 9$.

15. Reduce $\frac{2}{x} - \frac{3}{x} = \frac{a}{x} - c$. Ans. $x = \frac{a+1}{c}$.

16. Reduce $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 39$.

17. Reduce $\frac{2x}{a} - \frac{x}{b} + c = d$.

18. Reduce $(a - b)x + \frac{x}{c} = \frac{1}{2}$.

19. Reduce $x - \left(\frac{x}{2} - \frac{x}{3}\right) = 5$. Ans. $x = 6$.

20. Reduce $6 - \frac{2x + 1}{5} = x - 4$.

21. Reduce $2x - \frac{9x - 29}{4} = 18 - \frac{6x + 11}{5}$.
Ans. $x = 9$.

22. Reduce $\frac{117 - x}{4} = 3x + \frac{x - 95}{3}$.

23. Reduce $2x - \frac{3x - 1}{2} = 14 - \frac{7x - 2}{3}$. Ans. $x = 5$.

24. Reduce $6x + 7\frac{1}{4} - \frac{x}{2} = 9\frac{1}{4} - \frac{2x}{7} + \frac{11x}{2}$.

NOTE. — Before clearing of fractions, transpose $7\frac{1}{4}$ and unite it with $9\frac{1}{4}$; also transpose $-\frac{x}{2}$, and unite it with $\frac{11x}{2}$.

25. Reduce $4x + \frac{x + 6}{5} = 5 + \frac{11 + 11x}{3}$.

26. Reduce $\frac{x - 1}{2} - \frac{x - 9}{6} = 21 - \frac{x + 3}{6}$. Ans. $x = 39$.

27. Reduce $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = d$. Ans. $x = \frac{abcd}{bc + ac + ab}$.

28. Reduce $\frac{2x - 3}{3} - 6 = \frac{x - 6}{4} + 7$.

29. Reduce $\frac{x - 1}{6} = 6 - \frac{22 - x}{5} - \frac{3 + x}{5}$. Ans. $x = 7$.

30. Reduce $19 + \frac{2x - 22}{12} = \frac{3x - 75}{6} + \frac{284 - 4x}{3}$.

31. Reduce $\frac{4x + 5}{5} - \frac{5x - 5}{4} = \frac{x + 1}{6} - 1$.

Ans. $x = 5$.

32. Reduce $\frac{18 - 5x}{3} - \frac{3x + 3}{4} = 4x - 17 + \frac{5x + 3}{6}$.

33. Reduce $4x - \frac{x - 12}{3} + 5 = \frac{20x + 21}{4} - \frac{1}{4}$.

34. Reduce $\frac{a}{x} - \frac{b}{c} = \frac{d}{m}$. Ans. $x = \frac{acm}{bm + cd}$.

35. Reduce $\frac{x}{b} - \frac{ax}{c} = 1 - 3ac$.

36. Reduce $\frac{5x + 3}{2} + 6 - \frac{3x + 15}{4} = 4 + \frac{6x + 10}{4}$.

37. Reduce $3x - \frac{3x - 19}{2} - 8 = \frac{23 - x}{4} + \frac{5x - 38}{3} + 10$.

Ans. $x = 19$.

38. Reduce $\frac{13 - 3x}{10} - \frac{3x + 2}{5} = 7 - 6x + \frac{8x - 13}{5}$.

39. Reduce $\frac{4(x - 7)}{7} + \frac{3(x + 1)}{11} = \frac{7x - 17}{10} + \frac{x}{21}$.

40. Reduce $x - \frac{4x - 6}{5} + 3 = \frac{19 - 4x}{3} - \frac{5x - 6}{7} + \frac{7x + 8}{6}$.

41. Reduce $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} + \frac{x}{d} = m$.

42. Reduce $\frac{7x + 5}{7} + \frac{6x - 30}{7x - 7} = x + 1$.

NOTE.—Multiply by 7, transpose, and unite.

43. Reduce $2(3 + x) : 6x - 9 = 2 : 3$. Ans. $x = 6$.

44. Reduce $\frac{x}{6} + \frac{x}{5} : \frac{x + 7}{2x + 14} = 11 : \frac{1}{2}$.

45. Reduce $\frac{a}{x} - b : c + d = \frac{m}{x} : n$.

PROBLEMS

PRODUCING EQUATIONS OF THE FIRST DEGREE CONTAINING BUT ONE UNKNOWN QUANTITY.

107. The problems given in this Section must either contain but one unknown quantity, or the unknown quantities must be so related to one another that if one becomes known the others also become known.

108. With beginners the chief difficulty in solving a problem is in translating the statements or conditions of the problem from common to algebraic language; i. e. in preparing the data, and forming an equation in accordance with the given conditions.

1. If three times a certain number is added to one half and one third of itself, the sum is 115. What is the number?

SOLUTION.

Let $x =$ the number;

then by the conditions of the problem,

$$3x + \frac{x}{2} + \frac{x}{3} = 115$$

Clearing of fractions, $18x + 3x + 2x = 690$

Uniting terms, $23x = 690$

Dividing by 23, $x = 30$

VERIFICATION.

$$3 \times 30 + \frac{30}{2} + \frac{30}{3} = 115$$

$$115 = 115$$

In this problem there is but one unknown quantity, which we represent by x.

2. There are three numbers of which the first is 6 more than the second, and 11 less than the third; and their sum is 101. What are the numbers?

SOLUTION.

Let	$x =$ the first,	
then	$x - 6 =$ the second,	
and	$x + 11 =$ the third.	
Their sum,	$3x + 5 = 101$	
	$3x = 96$	
	$x = 32$, the first,	
	$x - 6 = 26$, the second,	
	$x + 11 = 43$, the third.	

In this problem there are three unknown quantities; but they are so related to one another that, if any one becomes known, the other two will be known.

VERIFICATION.

$$32 + 26 + 43 = 101$$
$$101 = 101$$

From these examples we deduce the following

GENERAL RULE.

Let x (or some one of the latter letters of the alphabet) represent the unknown quantity; or, if there is more than one unknown quantity, let x represent one, and find the others by expressing in algebraic form their given relations to the one represented by x.

With the data thus prepared form an equation in accordance with the conditions given in the problem.

Solve the equation.

The three steps may be briefly expressed thus: —

1st. Preparing the Data;

2d. Forming the Equation;

3d. Solving the Equation.

3. The sum of three numbers is 960; the first is one half of the second and one third of the third. What are the numbers? Ans. 160, 320, and 480.

4. Find two numbers whose difference is 18 and whose sum 112. Ans. 47 and 65.

5. A man being asked how much he gave for his horse said, that if he had given \$70 more than three times as much as it cost, he would have given \$445. How much did his horse cost him?

6. A man being asked how many sheep he had, replied that if he had as many more, and two thirds as many, and three fifths as many, he should have 8 more than three times as many as he had. How many sheep had he?

7. Divide \$575 between A and B in such a manner that B may have two thirds as much as A.

Ans. A's share, \$345;
B's " \$230.

8. A father divided his estate among his three children so that the eldest had \$1440 less than one half of the whole, the second \$500 more than one third of the whole, and the youngest \$250 more than one fourth of the whole. What was the value of the estate?

SOLUTION.

Let $x =$ whole estate.

Then $\frac{x}{2} - 1440 =$ share of the eldest,

$\frac{x}{3} + 500 =$ " " " second,

$\frac{x}{4} + 250 =$ " " " youngest,

Their sum $\frac{13x}{12} - 690 = x$, whole estate.

$$\frac{x}{12} = 690$$

$$x = 8280, \text{ whole estate.}$$

9. A gentleman meeting five poor persons, distributed \$7.50, giving to the second twice, to the third three times, to the fourth four times, and to the fifth five times as much as to the first. How much did he give to each?

10. Divide 795 into two such parts that the greater divided by 3 shall be equal to the less divided by 2.

NOTE.—To avoid fractions, let $3x =$ the greater and $2x =$ the less.

Ans. 477 and 318.

11. Divide a into two such parts that the greater divided by b shall be equal to the less divided by c.

SOLUTION.

Let $x =$ the greater,

then $a - x =$ the less.

And $$\frac{x}{b} = \frac{a-x}{c}$$

Clearing of fractions, $cx = ab - bx$

Transposing, $bx + cx = ab$

Dividing by $b + c$, $x = \frac{ab}{b+c}$, the greater,

$$a - x = a - \frac{ab}{b+c} = \frac{ac}{b+c}, \text{ the less.}$$

12. What number is that which, if multiplied by 7, and the product increased by eleven times the number, and this sum divided by 9, will give the quotient 6?

13. If to a certain number 55 is added, and the sum divided by 9, the quotient will be 5 less than one fifth of the number. What is the number? Ans. 125.

14. As A and B are talking of their ages, A says to B, "If one third, one fourth, and seven twelfths of my age are added to my age, the sum will be 8 more than twice my age." What was A's age?

15. A farmer having bought a horse kept him six weeks at an expense of $20, and then sold him for four fifths of the original cost, losing thereby $50. How much did he pay for the horse? Ans. $150.

16. A man left $18204, to be divided among his widow, three sons, and two daughters, in such a manner that the widow should have twice as much as a son, and each son as much as both daughters. What was the share of each?

17. If a certain number is divided by 9, the sum of the divisor, dividend, and quotient will be 89. What is the number? Ans. 72.

18. If a certain quantity is divided by a, the sum of the divisor, dividend, and quotient will be b. What is the quantity? Ans. $\frac{ab - a^2}{a + 1}$.

19. Verify the answer to the preceding problem.

20. A farmer mixed together corn, barley, and oats. In all there were 80 bushels, and the mixture contained two thirds as much corn as barley and one fifth as much barley as oats. How many bushels of each were there?

21. Three men, A, B, and C, built 572 rods of fence. A built 8 rods per day, B 7, and C 5. A worked one half as many days as B, and B one third as many as C. How many days did each work?

22. What number is as much greater than 340 as its third part is greater than 34? Ans. 459.

23. A man meeting some beggars gave 3 cents to each, and had 4 cents left. If he had undertaken to give 5 cents to each, he would have needed 6 cents to complete the distribution. How many beggars were there, and how much money did he have?

SOLUTION.

Let $x =$ the number of beggars;

then, according to the first statement,

$3x + 4 =$ the number of cents he had,

and, according to the second statement,

$5x - 6 =$ the number of cents he had.

Therefore,

$$5x - 6 = 3x + 4$$

$$2x = 10$$

$x = 5$, the number of beggars,

and $3x + 4 = 19$, the number of cents he had.

24. A boy wishing to distribute all his money among his companions gave to each 2 cents, and had 3 cents left; therefore, collecting it again, he began to give 3 cents to each, but found that in this case there was one who had received none, and another who had only 2 cents. How many companions, and how much money had he? Ans. 7 companions, and 17 cents.

25. What two numbers whose difference is 35 are to each other as 4 : 5?

26. A man being asked the hour, answered that three times the number of hours before noon was equal to three fifths of the number since midnight. What was the time of day?

SOLUTION

Let $x =$ the number of hours since midnight, i. e. the time; then $12 - x =$ the number of hours before noon.

Then $$36 - 3x = \frac{3x}{5}$$

Clearing of fractions, $$180 - 15x = 3x$$

Whence $$18x = 180$$
$$x = 10.$$ Ans. 10 o'clock.

27. A gains in trade \$300; B gains one half as much as A, plus one third as much as C; and C gains as much as A and B. What is the gain of B and C?

Ans. B's, \$375; C's, \$675.

28. What number is to 28 increased by one third of the number as 2 : 3? Ans. 24.

29. What number is that whose fifth part exceeds its sixth by 15?

30. Divide \$3740 into two parts which shall be in the ratio of 10 : 7.

31. Divide a into two parts which shall be in the ratio of $b : c$. Ans. $\frac{ab}{b+c}$ and $\frac{ac}{b+c}$.

32 What number is that the sum of whose fourth part, fifth part, and sixth part is 37 ?

33. What quantity is that the sum of whose third part, fifth part, and seventh part is a? Ans. $\frac{105\,a}{71}$.

34. A farmer sold 17 bushels of oats at a certain price, and afterward 12 bushels at the same rate; the second time he received 55 shillings less than the first. What was the price per bushel?

35. A certain number consists of two figures whose sum is 9; and if 27 is added to the number, the order of the figures will be inverted. What is the number?

SOLUTION.

Let $x =$ the left-hand figure;

then $9 - x =$ the right-hand figure.

As figures increase from right to left in a tenfold ratio,

$10\,x + (9 - x) = 9\,x + 9 =$ the number;

and when the order of the figures is inverted,

$10\,(9 - x) + x = 90 - 9\,x =$ the resulting number.

Therefore $9\,x + 9 + 27 = 90 - 9\,x$

Or $18\,x = 54$

Whence $x = 3$, the left-hand figure,

and $9 - x = 6$, the right-hand figure.

Ans. 36.

36. A certain number consists of three figures whose sum is 6, and the middle figure is double the left-hand figure; and if 198 is added to the number, the order of the figures will be inverted. What is the number?

Ans. 123.

37. Two men 90 miles apart travel towards each other till they meet. The first travels 5 miles an hour and the second 4. How many miles does each travel before they meet?

38. A man hired six laborers, to the first of whom he paid 75 cents a week more than to the second; to the second, 80 cents more than to the third; to the third, 60 cents more than to the fourth; to the fourth, 50 cents more than to the fifth; to the fifth, 40 cents more than to the sixth; and to all he paid $68.15 a week. What did he pay to each a week?

39. What number is that to which if 20 is added two thirds of the sum will be 80?

40. What number is that to which if a is added $\frac{b}{c}$ of the sum will be d?

Ans. $\frac{cd}{b} - a$.

41. A man spent one fourth of his life in Ireland, one fifth in England, and the rest, which was 33 years, in the United States. To what age did he live?

42. A post is one fifth in the mud, two sevenths in the water, and 18 feet above the water. How long is the post?

43. What number is that whose half is as much less than 40 as three times the number is greater than 156?

Ans. 56.

44. Two workmen received the same sum for their labor; but if one had received $15 less and the other $15 more, one would have received just four times as much as the other. What did each receive?

45. Of the trees on a certain lot of land five sevenths are oak, one fifth are chestnut, and there are 32 less walnut trees than chestnut. How many trees are there?

46. Divide 474 into two parts such that, if the greater part is divided by 7 and the less by 3, the first quotient shall be greater than the second by 12.

Ans. 357 and 117.

47. Two persons, A and B, have each an annual income of \$1500. A spends every year \$400 more than B, and at the end of five years the amount of their savings is \$6000. What does each spend annually?

Ans. A \$1100, and B \$700.

48. In a skirmish the number of men captured was 41 more, and the number killed 26 less than the number wounded; 45 men ran away; and the whole number engaged was four times the number wounded. How many men belonged to the skirmishing party? Ans. 240.

49. A and B have the same salary. A runs into debt every year a sum equal to one sixth of his salary, while B spends only three fourths of his; at the end of five years B has saved \$1000 more than enough to pay A's debt. What is the salary of each? Ans. \$2400.

50. A man lived single one third of his life: after having been married two years more than one eighth of his life, he had a daughter who died ten years after him, and whose age at her death was one year less than two thirds the age of her father at his death. What was the father's age at his death?

SOLUTION.

Let $x =$ his age;

then $\frac{x}{3} =$ his age at marriage,

$\frac{x}{3} + \frac{x}{8} + 2 =$ his age at daughter's birth,

and $x - \left(\frac{x}{3} + \frac{x}{8} + 2\right) =$ her age at his death.

Then $x - \frac{x}{3} - \frac{x}{8} - 2 + 10 = \frac{2x}{3} - 1$

Transposing and uniting, $-\frac{x}{8} = -9$

$x = 72$, the father's age.

51. Divide $864 among three persons so that A shall have as much as B and C together, and B $5 as often as C $11.

52. A father and son are aged respectively 32 and 8. How long will it be before the son will be just one half the age of the father?

53. A man's age was to that of his wife at the time of their marriage as 4 : 3, and seven years after, their ages were as 5 : 4. What was the age of each at the time of their marriage?

54. One fifth of a certain number minus one fourth of a number 20 less is 2. What is the number? Ans. 60.

55. There are two numbers which are to each other as $\frac{1}{2} : \frac{1}{3}$; but if 9 is added to each, they will be as $\frac{1}{5} : \frac{1}{6}$. What are the numbers? Ans. 9 and 6.

56. A person having spent $150 more than one third of his income had $50 more than one half of it left. What was his income?

57. A merchant sold from a piece of cloth a number of yards, such that the number sold was to the number left as 4 : 5; then he cut off for his own use 15 yards, and found that the number of yards left in the piece was to the number sold as 1 : 2. How many yards did the piece originally contain? Ans. 45.

58. Four places, A, B, C, and D, are in a straight line, and the distance from A to D is 126 miles. The distance from A to B is to the distance from B to C as 3 : 4, and one third the distance from A to B added to three fourths the distance from B to C is twice the distance from C to D. What is the distance from A to B, from B to C, and from C to D?

59. A laborer was hired for 40 days; for each day he wrought he was to receive $2.50, and for each day he was idle he was to forfeit $1.25. At the end of the time he received $58.75. How many days did he work?

Ans. 29.

60. A cask which held 44 gallons was filled with a mixture of brandy, wine, and water. There were 10 gallons more than one half as much wine as brandy, and as much water as brandy and wine. How many gallons were there of each?

61. Two persons, A and B, travelling each with \$80, meet with robbers who take from A \$5 more than twice what they take from B; then B finds he has \$26 more than twice what A has. How much is taken from each?

Ans. From A, \$69; from B, \$32.

62. Four persons, A, B, C, and D, entered into partnership with a capital of \$84816; of which B put in twice as much as A, C as much as A and B, and D as much as A, B, and C. How much did each put in?

63. In three cities, A, B, and C, 1188 soldiers are to be raised. The number of enrolled men in A is to that in B as 3 : 5; and the number in B to that in C as 8 : 7. How many soldiers ought each city to furnish?

Ans. A, 288; B, 480; C, 420.

64. Divide \$65 among five boys, so that the fourth may have \$2 more than the fifth and \$3 less than the third, and the second \$4 more than the third and \$5 less than the first.

65. A merchant bought two pieces of cloth, one at the rate of \$5 for 7 yards, and the other \$2 for 3 yards; the second piece contained as many times 3 yards as the first times 4 yards. He sold each piece at the rate of \$6 for 7 yards, and gained \$24 by the bargain. How many yards were there in each piece?

Ans. First, 84; second, 63.

66. A drover had the same number of cows and sheep. Having sold 17 cows and one third of his sheep, he finds he has three and a half times as many sheep as cows left. How many of each did he have at first?

67. A flour dealer sold one fourth of all the flour he had and one fourth of a barrel; afterward he sold one third of what he had left and one third of a barrel; and then one half of the remainder and one half of a barrel; and had 15 barrels left. How many had he at first?

SOLUTION.

Let $x =$ number at first;

then $\frac{3x}{4} - \frac{1}{4} =$ number after first sale,

$$\frac{2}{3}\left(\frac{3x}{4} - \frac{1}{4}\right) - \frac{1}{3} = \frac{x}{2} - \frac{1}{2} = \text{number after second sale},$$

and $\frac{1}{2}\left(\frac{x}{2} - \frac{1}{2}\right) - \frac{1}{2} = \frac{x}{4} - \frac{3}{4} =$ number after third sale.

Then $\frac{x}{4} - \frac{3}{4} = 15$

Clearing of fractions, $x - 3 = 60$

Whence $x = 63$, number at first.

68. A merchant bought a barrel of oil for \$50; at the same rate per gallon as he paid, he sold to one man 15 gallons; then to another at the same rate two fifths of the remainder for \$14. How many gallons did he buy in the barrel?

69. Two pieces of cloth of the same length but different prices per yard were sold, one for \$5 and the other for \$7.50. If there had been 5 more yards in each, at the same rate per yard as before, they would have come to \$15.47$\frac{13}{21}$. How many yards were there in each? Ans. 21.

70. A and B began trade with equal sums of money. The first year A lost one third of his money, and B gained \$750. The second year A doubled what he had at the end of the first year, and B lost \$150, when the two had again an equal sum. What did each have at first?

71. A man distributed among his laborers \$2.50 apiece, and had \$25 left. If he had given each \$3 as long as his money lasted, three would have received nothing. How many laborers were there, and how much money did he have? Ans. 68 laborers, and \$195.

72. A man who owned two horses bought a saddle for \$35. When the saddle was put on one horse, their value together was double the value of the other horse; but when the saddle was put on the other horse, their value together was four fifths of the value of the first horse. What was the value of each horse?

73. From a cask two thirds full 18 gallons were taken, when it was found to be five ninths full. How many gallons will the cask hold?

74. A farmer had two flocks of sheep, and sold one flock for \$60. Now a sheep of the flock sold was worth 4 of those left, and the whole value of those left was \$8 more than the price of 8 sheep of those sold, and the flock left contained 40 sheep. How many sheep did the farmer sell, and what was the value of a sheep of each flock?

Ans. Number sold, 15; value, \$4 and \$1.

75. A man has seven sons with 2 years between the ages of any two successive ones, and the sum of all their ages is ten times the age of the youngest. What is the age of each?

76. Divide 75 into two parts such that the greater increased by 9 shall be to the less diminished by 4 as $3:1$.

77. Divide a into two parts such that the greater increased by b shall be to the less diminished by c as $m:n$.

78. What two numbers are as $3:4$, while if 8 be added to each the sums will be as $5:6$?

79. Divide 127 into two parts, such that the difference between the greater and 130 shall be equal to five times the difference between the less and 63.

SECTION XIV.

EQUATIONS

OF THE FIRST DEGREE CONTAINING TWO UNKNOWN QUANTITIES.

109. INDEPENDENT EQUATIONS are such as cannot be derived from one another, or reduced to the same form.

Thus, $x + y = 10$, $\frac{x}{2} + \frac{y}{2} = 5$, and $4x + 3y = 40 - y$ are not independent equations, since any one of the three can be derived from any other one; or they can all be reduced to the form $x + y = 10$. But $x + y = 10$ and $4x = y$ are independent equations.

110. To find the value of several unknown quantities, there must be as many independent equations in which the unknown quantities occur as there are unknown quantities.

From the equation $x + y = 10$ we cannot determine the value of either x or y in known terms. If y is transposed, we have $x = 10 - y$; but since y is unknown, we have not determined the value of x. We may suppose y equal to any number whatever, and then x would equal the remainder obtained by subtracting y from 10. It is only required by the equation that the sum of two numbers shall equal 10; but there is an infinite number of pairs of numbers whose sum is equal to 10. But if we have also the equation $4x = y$, we may put this value of y in the first equation, $x + y = 10$, and obtain $x + 4x = 10$, or $x = 2$; then $4x = 8 = y$, and we have the value of each of the unknown quantities.

ELIMINATION.

111. ELIMINATION is the method of deriving from the given equations a new equation, or equations, containing one (or more) less unknown quantity. The unknown quantity thus excluded is said to be *eliminated.*

There are three methods of elimination: —

I. By substitution.

II. By comparison.

III. By combination.

CASE I.

112. Elimination by substitution.

1. Given $\left\{\begin{matrix} 4x + 5y = 23 \\ 5x + 4y = 22 \end{matrix}\right\}$, to find x and y.

OPERATION.

$$4x + 5y = 23 \quad (1) \qquad 5x + 4y = 22 \quad (2)$$

$$y = \frac{23 - 4x}{5} \quad (3) \qquad 5x + 4\left(\frac{23 - 4x}{5}\right) = 22 \quad (4)$$

$$25x + 92 - 16x = 110 \quad (5)$$

$$y = \frac{23 - 8}{5} = 3 \quad (7) \qquad x = 2 \quad (6)$$

Transposing $4x$ in (1) and dividing by 5, we have (3), which gives an expression for the value of y. Substituting this value of y in (2), we have (4), which contains but one unknown quantity; i. e. y has been eliminated. Reducing (4) we obtain (6), or $x = 2$. Substituting this value of x in (3), we obtain (7), or $y = 3$. Hence,

RULE.

Find an expression for the value of one of the unknown quantities in one of the equations, and substitute this value for the same unknown quantity in the other equation.

NOTE. — After eliminating, the resulting equation is reduced by the rule in Art. 102. The value of the unknown quantity thus found must be substituted in one of the equations containing the two unknown quantities, and this reduced by the rule in Art. 102.

Find the values of x and y in the following equations: —

2. Given $\left\{\begin{matrix} x + y = 17 \\ x - y = 3 \end{matrix}\right\}$. Ans. $\left\{\begin{matrix} x = 10. \\ y = 7. \end{matrix}\right.$

3. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=12 \\ \frac{x}{7}+\frac{y}{5}=\ 5\end{array}\right\}$. Ans. $\left\{\begin{array}{l}x=14. \\ y=15.\end{array}\right.$

4. Given $\left\{\begin{array}{r}2x-\frac{y}{2}-\ \ 3=0 \\ x+y-29=0\end{array}\right\}$. Ans. $\left\{\begin{array}{l}x=\ 7. \\ y=22.\end{array}\right.$

5. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=\frac{13}{36} \\ 2x+3y=2\end{array}\right\}$.

6. Given $\left\{\begin{array}{r}\frac{x}{2}-\frac{y}{3}=\ 0 \\ 3x-y=15\end{array}\right\}$.

CASE II.

113. Elimination by comparison.

1. Given $\left\{\begin{array}{r}x-2y=\ 6 \\ 2x-y=27\end{array}\right\}$, to find x and y.

OPERATION.

$$x-2y=6 \quad (1) \qquad 2x-y=27 \quad (2)$$

$$x=6+2y \quad (3) \qquad x=\frac{27+y}{2} \quad (4)$$

$$6+2y=\frac{27+y}{2} \quad (5)$$

$$12+4y=27+y \quad (6)$$

$$y=\ 5 \quad (7)$$

$$x=\ 6+10=16 \quad (8)$$

Finding an expression for the value of x from both (1) and (2), we have (3) and (4). Placing these two values of x equal to each other (Art. 13, Ax. 8), we form (5), which contains but one unknown quantity. Reducing (5) we obtain (7), or $y=5$. Substituting this value of y in (3), we have (8), or $x=16$. Hence,

RULE.

Find an expression for the value of the same unknown quantity from each equation, and put these expressions equal to each other.

By this method of elimination find the values of x and y in the following equations: —

2\. Given $\left\{\begin{array}{r} 3x - \frac{y}{3} = 2 \\ x + y = 4 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 1. \\ y = 3. \end{array}\right.$

3\. Given $\left\{\begin{array}{r} \frac{x}{2} + y = 12 \\ \frac{x}{3} - \frac{y}{2} = 8 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 24. \\ y = 0. \end{array}\right.$

4\. Given $\left\{\begin{array}{r} 3x + 5y = 2 \\ 2x + \frac{5}{3}y = 1 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \frac{1}{3}. \\ y = \frac{1}{5}. \end{array}\right.$

5\. Given $\left\{\begin{array}{r} \frac{x+y}{3} - \frac{x-y}{5} = 4 \\ \frac{x+y}{3} + \frac{x-y}{5} = 2 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 2. \\ y = 7. \end{array}\right.$

6\. Given $\left\{\begin{array}{r} 3(x-y) - 9 = 0 \\ \frac{y}{2} + \frac{x-y}{6} - 2 = 0 \end{array}\right\}$.

7\. Given $\left\{\begin{array}{r} 6x - 5y = 17 \\ 2x - y = 13 \end{array}\right\}$.

8\. Given $\left\{\begin{array}{r} \frac{x}{4} - \frac{y}{5} = 0 \\ \frac{x}{4} + \frac{y}{5} = 4 \end{array}\right\}$.

CASE III.

114. Elimination by combination.

1. Given $\left\{\begin{matrix} 3x - 2y = 7 \\ 2x - 3y = 3 \end{matrix}\right\}$, to find x and y.

OPERATION.

$3x - 2y = 7$	(1)	$2x - 3y = 3$	(2)
$6x - 4y = 14$	(3)		
$6x - 9y = 9$	(4)		
$5y = 5$	(5)	$2x - 3 = 3$	(7)
$y = 1$	(6)	$x = 3$	(8)

If we multiply (1) by 2, and (2) by 3, we have (3) and (4), in which the coefficients of x are equal; subtracting (4) from (3), we have (5), which contains but one unknown quantity. Reducing (5), we have (6), or $y = 1$; substituting this value of y in (2), we obtain (7), which reduced gives (8), or $x = 3$.

2. Given $\left\{\begin{matrix} \frac{x}{2} - \frac{y}{4} = 6 \\ \frac{x}{3} + \frac{y}{2} = 12 \end{matrix}\right\}$, to find x and y.

OPERATION.

$\frac{x}{2} - \frac{y}{4} = 6$	(1)	$\frac{x}{3} + \frac{y}{2} = 12$	(2)
		$x - \frac{y}{2} = 12$	(3)
$9 - \frac{y}{4} = 6$	(6)	$\frac{4x}{3} = 24$	(4)
$y = 12$	(7)	$x = 18$	(5)

If we multiply (1) by 2, we have (3), an equation in which y has the same coefficient as in (2); since the signs of y are different in (2) and (3), if we add these two equations together, we have (4), which contains but one unknown quantity. Reducing (4), we have (5), or $x = 18$. Substituting this value of x in (1), we have (6), which reduced gives (7), or $y = 12$. Hence,

RULE.

Multiply or divide the equations so that the coefficients of the quantity to be eliminated shall become equal; then, if the signs of this quantity are alike in both, subtract one equation from the other; if unlike, add the two equations together.

NOTE. — The least multiplier for each equation will be that which will make the coefficient of the quantity to be eliminated the least common multiple of the two coefficients of this quantity in the given equations. It is always best to eliminate that quantity whose coefficients can most easily be made equal.

By this method of elimination find the values of x and y in the following equations: —

3. Given $\left\{\begin{array}{l} 7x + 3y = 33 \\ 2x + 7y = 34 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 3. \\ y = 4. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 8x + 6y = 6 \\ 10x - 3y = 4 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \frac{1}{2}. \\ y = \frac{1}{3}. \end{array}\right.$

5. Given $\left\{\begin{array}{l} \frac{x}{3} + \frac{y}{4} = 12 \\ \frac{x}{9} + \frac{y}{6} = 5 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 27. \\ y = 12. \end{array}\right.$

6. Given $\left\{\begin{array}{l} \frac{x+y}{6} - 3 = 0 \\ \frac{x-y}{2} - 1 = 0 \end{array}\right\}$.

7. Given $\left\{\begin{array}{l} \frac{2}{3}x - \frac{3}{4}y = 0 \\ \frac{2}{5}x + \frac{5}{6}y = 10\frac{4}{15} \end{array}\right\}$.

115. Find the values of x and y in the following

EXAMPLES.

NOTE. — Which of the three methods of elimination should be used depends upon the relations of the coefficients to each other. That one which will eliminate the quantity desired with the least work is the best.

1. Given $\begin{cases} 2x + 3y = 25 \\ 7x + 2y = 28 \end{cases}$. Ans. $\begin{cases} x = 2. \\ y = 7. \end{cases}$

2. Given $\begin{cases} 5x - y = 0 \\ 7x + \frac{2}{3}y = 31 \end{cases}$. Ans. $\begin{cases} x = 3. \\ y = 15. \end{cases}$

3. Given $\begin{cases} \dfrac{x+y}{2} - y = 5 \\ \dfrac{x-2y}{3} - y = 2 \end{cases}$. Ans. $\begin{cases} x = 11. \\ y = 1. \end{cases}$

4. Given $\begin{cases} \dfrac{x-1}{2} + y = \dfrac{17}{4} \\ 2x - 4y = 17 \end{cases}$. Ans. $\begin{cases} x = 9. \\ y = \frac{1}{4}. \end{cases}$

5. Given $\begin{cases} \dfrac{2x+3y}{3} - \dfrac{1}{3} = \dfrac{x+2y+3}{2} \\ 7 - \dfrac{2x-2y}{3} = 3 \end{cases}$. Ans. $\begin{cases} x = 11. \\ y = 5. \end{cases}$

6. Given $\begin{cases} \dfrac{x}{7} + \dfrac{y}{3} = 6 \\ \dfrac{x}{8} + \dfrac{y}{16} = 2\frac{1}{2} \end{cases}$.

7. Given $\begin{cases} \dfrac{x+y}{3} - \dfrac{x-y}{3} = 18 \\ \dfrac{x-y}{11} - \dfrac{x+y}{7} = -8\frac{10}{11} \end{cases}$.

8. Given $\left\{\begin{array}{l} \frac{2x}{5} + \frac{5y}{3} = \frac{8}{15} \\ \frac{3}{4}x - y = \frac{7}{40} \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \frac{1}{2}. \\ y = \frac{1}{5}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} 5x + \frac{y}{5} = 54 \\ \frac{x}{5} + 5y = 102 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 10. \\ y = 20. \end{array}\right.$

10. Given $\left\{\begin{array}{l} \frac{x+y}{4} + \frac{y}{3} = \frac{2y-x}{2} + 24\frac{1}{3} \\ \frac{x}{3} - 8 = \frac{y}{7} + 4 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 48. \\ y = 28. \end{array}\right.$

11. Given $\left\{\begin{array}{l} \frac{x}{4} + \frac{y}{3} = 0 \\ 4x - 3y = 25 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 4. \\ y = -3. \end{array}\right.$

12. Given $\left\{\begin{array}{l} \frac{x-y}{b} = 2 \\ \frac{x+y}{a} = 2 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = a + b. \\ y = a - b. \end{array}\right.$

13. Given $\left\{\begin{array}{l} 1 - \frac{y-x}{10} = 0 \\ \frac{y}{4} - 3 = \frac{x}{5} \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 10. \\ y = 20. \end{array}\right.$

14. Given $\left\{\begin{array}{l} \frac{x+y}{4} = y - 2 \\ x - \frac{4y-4}{2} = 0 \end{array}\right\}$.

15. Given $\left\{\begin{array}{l} \frac{y}{2} - \frac{x}{7} + 2 = 5 \\ \frac{y}{5} - 1 = \frac{x}{2} - 6 \end{array}\right\}$.

16. Given $\left\{\begin{array}{l} y - 5 = \frac{x}{3} + 2 \\ \frac{y - x}{3} = x - \frac{4}{3} \end{array}\right\}$.

17. Given $\left\{\begin{array}{l} \frac{x + y}{3} = x - \frac{1}{3} \\ 4 - \frac{y - x}{4} = 3\frac{3}{4} \end{array}\right\}$.

18. Given $\left\{\begin{array}{l} 2y - \frac{x + 3}{4} = 6 \\ 3x - \frac{7 - y}{3} = 14 \end{array}\right\}$.

19. Given $\left\{\begin{array}{l} \frac{4x - 7y}{5} = x - 4 \\ \frac{3x + y}{4} = y + 3 \end{array}\right\}$.

PROBLEMS

PRODUCING EQUATIONS OF THE FIRST DEGREE CONTAINING TWO UNKNOWN QUANTITIES.

116. Many of the problems given in Section XIII. contain two or more unknown quantities; but in every case these are so related to each other that, if one becomes known, the others become known also; and therefore the problems can be solved by the use of a single letter. But many problems, on account of the complicated conditions, cannot be performed by the use of a single letter. No problem can be solved unless the conditions given are sufficient to form as many independent equations as there are unknown quantities.

1. A grocer sold to one man 7 apples and 5 pears for 41 cents; to another at the same rate 11 apples and 3 pears for 45 cents. What was the price of each?

SOLUTION.

Let $x =$ the price of an apple,
and $y =$ " " " a pear.

Then, by the conditions,

$$7x + 5y = 41 \quad (1) \qquad \text{and} \quad 11x + 3y = 45 \quad (2)$$

$$55x + 15y = 225 \quad (3)$$

$$21x + 15y = 123 \quad (4)$$

$$34x = 102 \quad (5)$$

$$x = 3 \quad (6)$$

$$21 + 5y = 41 \quad (7)$$

$$y = 4 \quad (8)$$

We multiply (2) by 5 and (1) by 3, and obtain (3) and (4); subtracting (4) from (3) we have (5), which reduced gives (6), or $x = 3$. Substituting this value of x in (1), we have (7), which reduced gives (8), or $y = 4$.

2. There is a fraction such that if 2 is added to the numerator the fraction will be equal to $\frac{1}{2}$; but if 3 is added to the denominator the fraction will be equal to $\frac{1}{3}$. What is the fraction?

SOLUTION.

Let $\frac{x}{y} =$ the fraction.

Then, by the conditions,

$$\frac{x+2}{y} = \frac{1}{2} \quad (1) \qquad \text{and} \quad \frac{x}{y+3} = \frac{1}{3} \quad (2)$$

$$3x = y + 3 \quad (3)$$

$$2x + 4 = y \quad (4)$$

$$x - 4 = 3 \quad (5)$$

$$x = 7 \quad (6)$$

$$14 + 4 = 18 = y \quad (7) \qquad \frac{x}{y} = \frac{7}{18} \quad (8)$$

Clearing (1) and (2) of fractions, we obtain (3) and (4); subtracting (4) from (3), we obtain (5), which reduced gives (6), or $x = 7$. Substituting this value of x in (4), we have (7), or $y = 18$. Hence, $\frac{x}{y} = \frac{7}{18}$

3. There are two numbers whose sum is 28, and one fourth of the first is 3 less than one fourth of the second. What are the numbers? Ans. 8 and 20.

4. The ages of two persons, A and B, are such that 5 years ago B's age was three times A's; but 15 years hence B's age will be double A's. What is the age of each? Ans. A's, 25; B's, 65.

5. There are two numbers such that one third of the first added to one eighth of the second gives 39; and four times the first minus five times the second is zero. What are the numbers?

6. Find a fraction such that if 6 is added to the numerator its value will be $\frac{1}{2}$, but if 3 be added to the denominator its value will be $\frac{1}{5}$? Ans. $\frac{5}{22}$.

7. What are the two numbers whose difference is to their sum as 1 : 2, and whose sum is to their product as 4 : 3?

SOLUTION.

Let $x =$ the greater and $y =$ the less.

Then $x - y : x + y = 1 : 2$ (1) $\qquad x + y : xy = 4 : 3$ (2)

$2x - 2y = x + y$ (3) $\qquad 3x + 3y = 4xy$ (4)

$x = 3y$ (5) $\qquad 9y + 3y = 12y^2$ (6)

$x = 3$ (7) $\qquad 1 = y$ (8)

Having written (1) and (2) in accordance with the statement in the problem, we form from them (3) and (4) by Art. 106. Reducing (3), we obtain (5); substituting this value of x in (4), we have (6), which, though an equation of the second degree, can be at once reduced to an equation of the first degree by dividing each term by y; performing this division and reducing, we obtain (8) or $y = 1$; substituting this value of y in (5) we obtain (7), or $x = 3$.

8. What are the two numbers whose difference is to their sum as 3 : 20, and three times the greater minus twice the less is 35?

9. There is a number consisting of two figures, which is seven times the sum of its figures; and if 36 is subtracted from it, the order of the figures will be inverted. What is the number? Ans. 84.

10. There is a number consisting of two figures, the first of which is the greater; and if it is divided by the sum of its figures, the quotient is 6; and if the order of the figures is inverted, and the resulting number divided by the difference of its figures plus 4, the quotient will be 9. What is the number? Ans. 54.

11. As John and James were talking of their money, John said to James, "Give me 15 cents, and I shall have four times as much as you will have left." James said to John, "Give me $7\frac{1}{2}$ cents, and I shall have as much as you will have left." How many cents did each have? Ans. John, 45 cents; James, 30 cents.

12. The height of two trees is such that one third of the height of the shorter added to three times that of the taller is 360 feet; and if three times the height of the shorter is subtracted from four times that of the taller, and the remainder divided by 10, the quotient is 17. Required the height of each tree.

Ans. 90 and 110 feet.

13. A farmer who had $41 in his purse gave to each man among his laborers $2.50, to each boy $1, and had $15 left. If he had given each man $4 and then each boy $3 as long as his money lasted, 3 boys would have received nothing. How many men and how many boys did he hire?

14. A man worked 10 days and his son 6, and they received \$31; at another time he worked 9 days and his son 7, and they received \$29.50. What were the wages of each?

15. A said to B, "Lend me one fourth of your money, and I can pay my debts." B replied, "Lend me \$100 less than one half of yours, and I can pay mine." Now A owed \$1200 and B \$1900. How much money did each have in his possession?

Ans. A, \$800; B, \$1600.

16. If a is added to the difference of two quantities, the sum is b; and if the greater is divided by the less, the quotient will be c. What are the quantities?

Ans. $\dfrac{bc - ac}{c - 1}$ and $\dfrac{b - a}{c - 1}$.

17. A man owns two pieces of land. Three fourths of the area of the first piece minus two fifths of the area of the second is 12 acres; and five eighths of the area of the first is equal to four ninths of the area of the second. How many acres are there in each?

Ans. 1st, 64 acres; 2d, 90 acres.

18. A and B begin business with different sums of money; A gains the first year \$350, and B loses \$500, and then A's stock is to B's as 9 : 10 If A had lost \$500 and B gained \$350, A's stock would have been to B's as 1 : 3. With what sum did each begin?

Ans. A, \$1450; B, \$2500.

19. If a certain rectangular field were 4 feet longer and 6 feet broader, it would contain 168 square feet more; but if it were 6 feet longer and 4 feet broader, it would contain 160 square feet more. Required its length and breadth.

20. A market-man bought eggs, some at 3 for 7 cents and some at 2 for 5 cents, and paid for the whole $2.62; he afterward sold them at 36 cents a dozen, clearing $0.62. How many of each kind did he buy?

21. A and B can perform a piece of work together in 12 days. They work together 7 days, and then A finishes the work alone in 15 days. How long would it take each to do the work? Ans. A 36 and B 18 days.

22. "I was ten times as old as you 12 years ago," said a father to his son; "but 3 years hence I shall be only two and one half times as old as you." What was the age of each?

23. If 3 is added to the numerator of a certain fraction, its value will be $\frac{3}{5}$; and if 4 is subtracted from the denominator, its value will be $\frac{1}{2}$. What is the fraction?

24. A farmer sold to one man 7 bushels of oats and 5 bushels of corn for $12.76, and to another, at the same rate, 5 bushels of oats and 7 bushels of corn for $13.40. What was the price of each?

25. Find two quantities such that one third of the first minus one half the second shall equal one sixth of a; and one fourth of the first plus one fifth of the second shall equal one half of a. Ans. $\frac{34a}{23}$ and $\frac{15a}{23}$.

26. A person had a certain quantity of wine in two casks. In order to obtain an equal quantity in each, he poured from the first into the second as much as the second already contained; then he poured from the second into the first as much as the first then contained; and, lastly, he poured from the first into the second as much as the second still contained; and then he had 16 gallons in each cask. How many gallons did each originally contain? Ans. 1st, 22; 2d, 10 gallons.

SECTION XV.

EQUATIONS

OF THE FIRST DEGREE CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

117. The methods of elimination given for solving equations containing two unknown quantities apply equally well to those containing more than two unknown quantities.

1. Given $\left\{\begin{array}{r} x+y-z=4 \\ 2x+3y+4z=17 \\ 3x-2y+5z=5 \end{array}\right\}$, to find x, y, and z.

OPERATION.

$$\begin{array}{lllll}
x+y-z=4 \quad (1) & & 2x+3y+4z=17 \quad (2) & & 3x-2y+5z=5 \quad (3) \\
 & & \underline{2x+2y-2z=8} \quad (4) & & \underline{3x+3y-3z=12} \quad (5) \\
 & & y+6z=9 \quad (6) & & 5y-8z=7 \quad (7) \\
 & & & & \underline{5y+30z=45} \quad (8) \\
x+3-1=4 \quad (13) & & y+6=9 \quad (11) & & 38z=38 \quad (9) \\
x=2 \quad (14) & & y=3 \quad (12) & & z=1 \quad (10)
\end{array}$$

Multiplying equation (1) by 2 gives equation (4), which we subtract from (2), and obtain (6); multiplying (1) by 3 gives (5), and subtracting (5) from (3) gives (7). We have now obtained two equations, (6) and (7), containing but two unknown quantities. Multiplying (6) by 5, we obtain (8), and subtracting (7) from (8), we obtain (9), which reduced gives $z=1$. Substituting this value of z in (6), and reducing, we obtain $y=3$. Substituting these values of y and z in (1), and reducing, we obtain $x=2$.

2. Given $\left\{\begin{array}{l} x+y=26 \\ y+z=29 \\ z+w=56 \\ w+u=81 \\ u+x=46 \end{array}\right\}$, to find u, w, x, y, and z.

OPERATION.

$$\begin{array}{lllll}
x+y=26 \quad (1) & y+z=29 \quad (2) & z+w=56 \quad (3) & w+u=81 \quad (4) & u+x=46 \quad (5) \\
 & \underline{y+x=26} & \underline{z-x=\ 3} & \underline{w+x=53} & \underline{u-x=28} \\
 & z-x=\ 3 \quad (6) & w+x=53 \quad (7) & u-x=28 \quad (8) & 2x=18 \quad (9) \\
y=17 \quad (11) & z=12 \quad (12) & w=44 \quad (13) & u=37 \quad (14) & x=\ 9 \quad (10)
\end{array}$$

Here we subtract (1) from (2), and obtain (6); then (6) from (3), and obtain (7); then (7) from (4), and obtain (8); then (8) from (5), and obtain (9), which reduced gives (10), or $x = 9$. Substituting this value of x in (1), (6), (7), and (8), and reducing, we obtain (11), (12), (13), and (14), or $y = 17$, $z = 12$, $w = 44$, and $u = 37$.

Hence, for solving equations containing any number of unknown quantities,

RULE.

From the given equations deduce equations one less in number, containing one less unknown quantity; and continue thus to eliminate one unknown quantity after another, until one equation is obtained containing but one unknown quantity. Reduce this last equation so as to find the value of this unknown quantity; then substitute this value in an equation containing this and but one other unknown quantity, and reducing the resulting equation, find the value of this second unknown quantity; substitute again these values in an equation containing no more than these two and one other unknown quantity, and reduce as before; and so continue, till the value of each unknown quantity is found.

NOTE. — The process can often be very much abridged by the exercise of judgment in selecting the quantity to be eliminated, the equations from which the other equations are to be deduced, the method of elimination which shall be used, and the simplest equations in which to substitute the values of the quantities which have been found.

Find the values of the unknown quantities in the following equations : —

3. Given $\left\{\begin{array}{r} x+y+z+w=16 \\ y+z+w+u=18 \\ x+z+w+u=17 \\ x+y+w+u=14 \\ x+y+z+u=15 \end{array}\right\}$.

NOTE. — If these equations are added together and the sum divided by 4, we shall have $x+y+z+w+u=20$; and if from this the given equations are successively subtracted, the values of the unknown quantities become known.

Ans. $\left\{\begin{array}{l} x=2. \\ y=3. \\ z=6. \\ u=4. \\ w=5. \end{array}\right.$

4. Given $\left\{\begin{array}{r} x+y+\frac{1}{2}z=9 \\ x+3y+2z=26 \\ 3x+\frac{1}{2}y+\frac{1}{3}z=10 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=2. \\ y=4. \\ z=6. \end{array}\right.$

5. Given $\left\{\begin{array}{r} 2x+3y+4z=67 \\ \frac{x}{2}+\frac{y}{2}=4 \\ 2y+z=25 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=1. \\ y=7. \\ z=11. \end{array}\right.$

6. Given $\left\{\begin{array}{r} x-y-z=1 \\ x+2y-10z=1 \\ 2x-4y+3z=1 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=5. \\ y=3. \\ z=1. \end{array}\right.$

7. Given $\left\{\begin{array}{r} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=22 \\ \frac{1}{4}x+\frac{1}{4}y+\frac{1}{2}z=24 \\ \frac{1}{5}x+\frac{1}{6}y+\frac{1}{8}z=10 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=20. \\ y=12. \\ z=32. \end{array}\right.$

8. Given $\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{5}{6} \\ \frac{1}{y}+\frac{1}{z}=\frac{7}{12} \\ \frac{1}{x}+\frac{1}{z}=\frac{3}{4} \end{array}\right\}$.

Note. — The best method for this example is that used in Example 3, without clearing of fractions.

9. Given $\left\{\begin{array}{r} x+y+z=6 \\ 2x+3y+4z=20 \\ 3x+7y+5z=32 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=1. \\ y=2. \\ z=3. \end{array}\right.$

10. Given $\left\{\begin{array}{l} x+\frac{1}{2}y=37 \\ y+\frac{1}{3}z=17 \\ z+\frac{1}{4}x=29 \end{array}\right\}$.

11. Given $\left\{\begin{array}{l} x+y=a \\ y+z=b \\ x+z=c \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\frac{1}{2}(a-b+c). \\ y=\frac{1}{2}(a+b-c). \\ z=\frac{1}{2}(b+c-a). \end{array}\right.$

12. Given $\left\{\begin{array}{l} abx+aby=a+b \\ acx+acz=a+c \\ bcy+bcz=b+c \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x=\frac{1}{a}. \\ y=\frac{1}{b}. \\ z=\frac{1}{c}. \end{array}\right.$

13. Given $\left\{\begin{array}{l} y+\frac{1}{2}x+\frac{1}{2}z=18 \\ x+27=\frac{1}{3}y+28 \\ 9x=2y \end{array}\right\}$.

PROBLEMS

PRODUCING EQUATIONS OF THE FIRST DEGREE CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

118. 1. A merchant has three kinds of flour. He can sell 1 bbl. of the first, 2 of the second, and 3 of the third for $85; 2 of the first, 1 of the second, and ½ bbl. of the third for $45.50; and 1 of each kind for $41. What is the price per bbl. of each?

Ans. 1st, $12; 2d, $14; 3d, $15.

2. Three boys, A, B, and C, divided a sum of money among themselves in such a manner that A and B received 18 cents, B and C 14 cents, and A and C 16. How much did each receive? Ans. A, 10; B, 8; C, 6 cents.

3. As three persons, A, B, and C, were talking of their ages, it was found that the sum of one half of A's age, one third of B's, and one fourth of C's was 33; that the sum of A's and B's was 13 more than C's age; while the sum of B's and C's was 3 less than twice A's age. What was the age of each? Ans. A's, 32; B's, 21; C's, 40.

4. As three drovers were talking of their sheep, says A to B, "If you will give me 10 of yours, and C one fourth of his, I shall have 6 more than C now has." Says B to C, "If you will give me 25 of yours, and A one fifth of his, I shall have 8 more than both of you will have left." Says C to A and B, "If one of you will give me 10, and the other 9, I shall have just as many as both of you will have left." How many did each have?

5. Divide 32 into four such parts that if the first part is increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, the sum, difference, product, and quotient shall all be equal.

Ans. 3, 9, 2, and 18.

6. If A and B can perform a piece of work together in $8\frac{2}{11}$ days, B and C in $9\frac{9}{13}$ days, and A and C in $8\frac{3}{4}$ days, in how many days can each do it alone?

Ans. A in 15, B in 18, and C in 21 days.

7. Find three numbers such that one half of the first, one third of the second, and one fourth of the third shall together be 56; one third of the first, one fourth of the second, and one fifth of the third, 43; one fourth of the first, one fifth of the second, and one sixth of the third, 35.

8. The sum of the three figures of a certain number is 12; the sum of the last two figures is double the first; and if 297 is added to the number, the order of its figures will be inverted. What is the number?

Ans. 417.

9. A man sold his horse, carriage, and harness for \$450. For the horse he received \$25 less than five times what he received for the harness; and one third of what he received for the horse was equal to what he received for the harness plus one seventh of what he received for the carriage. What did he receive for each?

Ans. Horse, \$225; carriage, \$175; harness, \$50.

10. A man owned three horses, and a saddle which was worth \$45. If the saddle is put on the first horse, the value of both will be \$30 less than the value of the second; if the saddle is put on the second horse, the value of both will be \$55 less than the value of the third; and if the saddle is put on the third horse, the value of both will be equal to twice the value of the second minus \$10 more than one fifth of the value of the first. What is the value of each horse?

Ans. 1st, \$100; 2d, \$175; 3d, \$275.

11. The sum of the numerators of two fractions is 7, and the sum of their denominators 16; moreover the sum of the numerator and denominator of the first is equal

to the denominator of the second; and the denominator of the second, minus twice the numerator of the first, is equal to the numerator of the second. What are the fractions? Ans. $\frac{2}{7}$ and $\frac{5}{9}$.

12. A man bought a horse, a wagon, and a harness, for \$180. The horse and harness cost three times as much as the wagon, and the wagon and harness one half as much as the horse. What was the cost of each?

13. A gentleman gives \$600 to be divided among three classes in such a way that each one of the best class is to receive \$10, and the remainder to be divided equally among those of the other two classes. If the first class proves to be the best, each one of the other two classes will receive \$5; if the second class proves to be the best, each one of the other two classes will receive $\$4\frac{2}{7}$; but if the third class proves to be the best, each one of the other two classes will receive \$2. What is the number in each class?

14. A cistern has 3 pipes opening into it. If the first should be closed, the cistern would be filled in 20 minutes; if the second, in 25 minutes; and if the third, in 30 minutes. How long would it take each pipe alone to fill the cistern, and how long would it take the three together?

Ans. 1st, $85\frac{5}{7}$ minutes; 2d, $46\frac{2}{13}$ minutes; 3d, $35\frac{5}{17}$ minutes. The three together, $16\frac{8}{37}$ minutes.

15. Three men, A, B, and C, had together \$24. Now if A gives to B and C as much as they already have and then B gives to A and C as much as they have after the first distribution, and again C gives to A and B as much as they have after the second distribution, they will all have the same sum. How much did each have at first? Ans. A, \$13; B, \$7, and C, \$4.

SECTION XVI.

POWERS AND ROOTS.

119. A POWER of any quantity is the product obtained by taking that quantity any number of times as a factor; and the *exponent* shows how many times the quantity is taken (Art. 24). Thus,

$a = a^1$ is the first power of a;
$a\,a = a^2$ " second power, or square, of a;
$a\,a\,a = a^3$ " third power, or cube, of a;
$a\,a\,a\,a = a^4$ " fourth power of a;

and so on.

120. In order to explain the use of negative indices, we form, by the rules of division, the following series:—

$$a^5,\ a^4,\ a^3,\ a^2,\ a,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \frac{1}{a^5},$$
$$a^5,\ a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ a^{-5}.$$

We form the first series as follows: a^5 divided by a gives a^4; a^4 by a, gives a^3; a^3 by a, gives a^2; a^2 by a, gives a; a by a, gives 1; 1 by a, gives $\frac{1}{a}$; $\frac{1}{a}$ by a, gives $\frac{1}{a^2}$; $\frac{1}{a^2}$ by a, gives $\frac{1}{a^3}$, and so on. The second series is formed in the same way from a^5 to a; but if we follow the same rule of division from a toward the right as from a^5 to a, viz. *subtracting the index of the divisor from that of the dividend*, a divided by a, gives a^0; a^0 by a, gives a^{-1}; read a, with the negative index one; a^{-1} by a, gives a^{-2}; a^{-2} by a, gives a^{-3}; and so on.

From this we learn,

1st. *That the* 0 *power of every quantity is* 1;

2d. *That* a^{-1}, a^{-2}, a^{-3}, *&c., are only different ways of writing* $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, *&c.*

Any two quantities at equal distances on opposite sides of a^0, or 1, are reciprocals of each other.

121. The rules given for the multiplication and division of powers of the same quantity (Arts. 50 and 54) apply equally well whether the exponents are positive or negative. For

$$a^5 \times a^{-3} = a^5 \times \frac{1}{a^3} = \frac{a^5}{a^3} = a^2$$

$$a^6 \div a^{-2} = a^6 \div \frac{1}{a^2} = a^6 \times a^2 = a^8$$

$$a^{-7} \div a^4 = \frac{1}{a^7} \div a^4 = \frac{1}{a^{11}}, \text{ or } a^{-11}$$

The following examples in multiplication are to be done according to the rules for the multiplication of powers of the same quantity by each other, given in Art. 50; and those in division, by the rule for the division of powers of the same quantity by each other, given in Art. 54.

1. Multiply x^7 by x^{-1}. Ans. x^6.
2. Multiply a^8 by a^{-5}.
3. Multiply x^6 by x^{-6}. Ans. x^0, or 1.
4. Multiply y^{-7} by y^4.
5. Multiply $a^{-4}x^2y^5$ by $a^{-1}x^{-2}y^2$.
   Ans. $a^{-5}x^0y^7$, or $a^{-5}y^7$.
6. Multiply $4x^{-8}y^{-5}z$ by $3x^8y^5z^4$.
7. Multiply $17x^2y^4z^{-8}$ by $4x^{-1}y^{-4}z^{-3}$.
8. Multiply $\frac{a^{-4}bc^{-7}}{5}$ by $5a^4b^{-1}c^7$.
9. Divide x^8 by x^{-3}. Ans. x^{11}.
10. Divide x^8 by x^{-7}.
11. Divide x^{-6} by x^{-2}. Ans. x^{-4}.

12. Divide y^{-5} by y^2.

13. Divide y^{-7} by y^{-9}. Ans. y^2.

14. Divide $a^{-3}bc^2$ by $a^2b^{-4}c^{-1}$. Ans. $a^{-5}b^5c^3$.

15. Divide $16x^2y^{-4}z$ by $4x^2y^{-4}z^3$.

16. Divide $4x^2y^{-3}z$ by $2ax^{-2}y^{-3}z^2$.

17. Divide $7a^2bx^{-4}y^3$ by $10ab^{-1}x^4y^{-3}z^2$.

18. Divide $144a^2bc^{-2}x^2y^{-7}z$ by $16a^2b^{-1}c^{-2}x^2y^7$.

122. It follows from the preceding article that *a factor may be transferred from the numerator of a fraction to its denominator, or vice versa, provided the sign of the exponent of the factor is changed from + to —, or — to +.* For

$$\frac{a^6}{x^4} = a^6 \times \frac{1}{x^4} = a^6 \times x^{-4} = a^6x^{-4}$$

$$\frac{a}{x^{-7}} = \frac{a}{\frac{1}{x^7}} = a \div \frac{1}{x^7} = a \times x^7 = ax^7$$

$$\frac{x^5}{y} = \frac{1}{y} \times x^5 = \frac{1}{y} \div \frac{1}{x^5} = \frac{1}{y} \div x^{-5} = \frac{1}{x^{-5}y}$$

$$\frac{a^{-7}}{x} = \frac{1}{a^7} \times \frac{1}{x} = \frac{1}{a^7x}$$

1. Transfer the denominator of $\frac{a^7x^4}{bc^2y^{-1}}$ to the numerator.

Ans. $a^7b^{-1}c^{-2}x^4y$.

2. Transfer the numerator of $\frac{a^{-2}b^7c^6}{x^4y^3z^{-1}}$ to the denominator.

Ans. $\frac{1}{a^2b^{-7}c^{-6}x^4y^3z^{-1}}$.

3. Transfer the denominator of $\frac{x^3yz^6}{a^{-2}bc^3}$ to the numerator.

4. Transfer the numerator of $\frac{ax^3y^{-4}z}{bcd}$ to the denominator.

5. Free from negative exponents $\dfrac{4\,a^{-1}\,b^3\,c^{-6}}{7\,d^{-7}\,x\,y\,z^{-4}}$.

Ans. $\dfrac{4\,b^3\,d^7\,z^4}{7\,a\,c^6\,x\,y}$.

6. Free from negative exponents $\dfrac{a^{-1}\,b^{-2}\,c}{x^{-3}\,y^6\,z^{-4}}$.

7. Free from negative exponents $\dfrac{a\,(x+y)^{-1}}{x-y}$.

Ans. $\dfrac{a}{x^2-y^2}$

8. Free from negative exponents $\dfrac{(x-y)\,(x+y)^{-1}}{(x-y)^{-2}\,(x+y)}$.

Ans. $\dfrac{(x-y)^3}{(x+y)^2}$.

9. Free from negative exponents $\dfrac{x^{-n}\,y^m}{a^{-1}\,b^{-n}\,z^m}$.

INVOLUTION.

123. Involution is the process of raising a quantity to a power.

124. A quantity is involved by taking it as a factor as many times as there are units in the index of the required power.

125. According to Art. 48,

$$(+a)\times(+a)=+a^2,$$
$$(+a)\times(+a)\times(+a)=(+a^2)\times(+a)=+a^3,$$

and so on;

and
$$(-a)\times(-a)=+a^2,$$
$$(-a)\times(-a)\times(-a)=(+a^2)\times(-a)=-a^3,$$
$$(-a)\times(-a)\times(-a)\times(-a)=(-a^3)\times(-a)=+a^4,$$

and so on.

Hence, for the signs we have the following

RULE.

Of a positive quantity all the powers are positive.

Of a negative quantity the even powers are positive, and the odd powers negative.

INVOLUTION OF MONOMIALS.

126. To raise a monomial to any required power.

1. Find the third power of $2a^3b$.

OPERATION.

$$(2a^3b)^3 = 2a^3b \times 2a^3b \times 2a^3b \quad (1)$$
$$= 2.\ 2.\ 2.\ a^3\, a^3\, a^3\, b\, b\, b \quad (2)$$
$$= 8a^9b^3 \quad (3)$$

According to Art. 124, to raise $2a^3b$ to the third power we take it as a factor three times (1); and as it makes no difference in the product in what order the factors are taken, we arrange them as in (2); performing the multiplication (Art. 50) expressed in (2), we have (3). Hence,

RULE.

Multiply the exponent of each letter by the index of the required power, and prefix the required power of the numerical coefficient, remembering that the odd powers of a negative quantity are negative, while all other powers are positive.

NOTE.—It follows that *the power of the product is equal to the product of the powers.*

2. Find the square of $2x$. Ans. $4x^2$.
3. Find the cube of $3x^2$. Ans. $27x^6$.
4. Find the fourth power of a^3b^2. Ans. $a^{12}b^8$.
5. Find the third power of $4a^2x$. Ans. $64a^6x^3$.
6. Find the square of $2x^{-1}$. Ans. $4x^{-2}$.
7. Find the cube of $3x^{-3}y^2$. Ans. $27x^{-9}y^6$.
8. Find the mth power of ab. Ans. $a^m b^m$.
9. Find the third power of $-3a^2b$. Ans. $-27a^6b^3$.

10. Expand $(-2a^3x)^4$. Ans. $16a^{12}x^4$.

11. Expand $(3a^2b^n)^m$. Ans. $3^m a^{2m} b^{mn}$.

12. Expand $(2x^2y)^2$.

13. Expand $(-4a^2x^3)^5$.

14. Expand $(-3x^3y)^3$.

15. Expand $(-a^{-2})^3$. Ans. $-a^{-6}$.

16. Expand $(x^{-3}y^2)^5$.

17. Expand $(-4x^{-3}y)^2$. Ans. $\frac{16y^2}{x^6}$.

18. Expand $(3a^mx^2)^4$.

19. Expand $(-2x^{-3}y^{-n})^3$.

20. Expand $(-3x^{-2n}y^m)^5$.

21. Expand $(-9a^{-2}b^{-m}x^2y^n)^2$.

INVOLUTION OF FRACTIONS.

127. To involve a fraction.

1. Find the cube of $\frac{a^2}{2b}$.

OPERATION.

$$\left(\frac{a^2}{2b}\right)^3 = \frac{a^2}{2b} \times \frac{a^2}{2b} \times \frac{a^2}{2b} = \frac{a^6}{8b^3}$$

According to Art. 124, to find the cube of any quantity we must take it three times as a factor; taking $\frac{a^2}{2b}$ three times as a factor, and performing the multiplication by Art. 95, we have $\frac{a^6}{8b^3}$. Hence,

RULE.

Involve both numerator and denominator to the required power.

2. Find the square of $\frac{2a}{3x^2}$. Ans. $\frac{4a^2}{9x^4}$.

3. Find the cube of $-\frac{ab}{3c^2d}$. Ans. $-\frac{a^3b^3}{27c^6d^3}$.

4. Find the fourth power of $\frac{2bc^2}{3x^{-m}}$.

5. Find the fifth power of $-\frac{x^2y^{-1}}{2ab^2}$.

6. Find the third power of $\frac{-2ax^{-1}}{5cy^{-m}}$.

7. Find the mth power of $\frac{a^2b^{-1}}{3x^{-2}y^n}$.

8. Find the fourth power of $\frac{2a^2x^{-1}}{-b^{-2}cd^n}$.

9. Find the third power of $-\frac{3a^3b^{-1}c^{-2}}{5x^my^2z^{-4}}$.

10. Find the fourth power of $-\frac{4x^3y^{-2}z^{-3}}{5m^{-2}n^{-2}p}$.

11. Find the fifth power of $-\frac{2x^{-1}y^2z}{3ab^{-2}c^3}$.

INVOLUTION OF BINOMIALS.

128. A BINOMIAL can be raised to any power by successive multiplications. But when a high power is required, the operation is long and tedious. The BINOMIAL THEOREM, first developed by Sir Isaac Newton, enables us to expand a binomial to any power by a short and speedy process.

129. In order to investigate the law which governs the expansion of a binomial we will expand $a + b$ and $a - b$ to the fifth power by multiplication.

$$
\begin{array}{llllllll}
a & +b & & & & & & \\
a & +b & & & & & & \\
\hline
a^2 & +ab & & & & & & \\
 & +ab & +b^2 & & & & & \\
\hline
a^2 & +2ab & +b^2 & & & & & \text{. . . . . . 2d power.} \\
a & +b & & & & & & \\
\hline
a^3 & +2a^2b & +ab^2 & & & & & \\
 & +a^2b & +2ab^2 & +b^3 & & & & \\
\hline
a^3 & +3a^2b & +3ab^2 & +b^3 & & & & \text{. . . . 3d power.} \\
a & +b & & & & & & \\
\hline
a^4 & +3a^3b & +3a^2b^2 & +ab^3 & & & & \\
 & +a^3b & +3a^2b^2 & +3ab^3 & +b^4 & & & \\
\hline
a^4 & +4a^3b & +6a^2b^2 & +4ab^3 & +b^4 & & & \text{. . 4th power.} \\
a & +b & & & & & & \\
\hline
a^5 & +4a^4b & +6a^3b^2 & +4a^2b^3 & +ab^4 & & & \\
 & +a^4b & +4a^3b^2 & +6a^2b^3 & +4ab^4 & +b^5 & & \\
\hline
a^5 & +5a^4b & +10a^3b^2 & +10a^2b^3 & +5ab^4 & +b^5 & & \text{. 5th power.}
\end{array}
$$

$$
\begin{array}{llllllll}
a & -b & & & & & & \\
a & -b & & & & & & \\
\hline
a^2 & -ab & & & & & & \\
 & -ab & +b^2 & & & & & \\
\hline
a^2 & -2ab & +b^2 & & & & & \text{. . . . . . 2d power.} \\
a & -b & & & & & & \\
\hline
a^3 & -2a^2b & +ab^2 & & & & & \\
 & -a^2b & +2ab^2 & -b^3 & & & & \\
\hline
a^3 & -3a^2b & +3ab^2 & -b^3 & & & & \text{. . . . 3d power.} \\
a & -b & & & & & & \\
\hline
a^4 & -3a^3b & +3a^2b^2 & -ab^3 & & & & \\
 & -a^3b & +3a^2b^2 & -3ab^3 & +b^4 & & & \\
\hline
a^4 & -4a^3b & +6a^2b^2 & -4ab^3 & +b^4 & & & \text{. . 4th power.} \\
a & -b & & & & & & \\
\hline
a^5 & -4a^4b & +6a^3b^2 & -4a^2b^3 & +ab^4 & & & \\
 & -a^4b & +4a^3b^2 & -6a^2b^3 & +4ab^4 & -b^5 & & \\
\hline
a^5 & -5a^4b & +10a^3b^2 & -10a^2b^3 & +5ab^4 & -b^5 & & \text{5th power.}
\end{array}
$$

By examining the different powers of $a+b$ and $a-b$ in these Examples, we shall find the following invariable laws governing the expansion :—

1*st. The leading quantity* (*i. e. the first quantity of the binomial*) *begins in the first term of the power with an exponent equal to the index of the power, and its exponent decreases regularly by one in each successive term till it disappears; the following quantity* (*i. e. the second quantity of the binomial*) *begins in the second term of the power with the exponent one, and its exponent increases regularly by one till in the last term it becomes the same as the index of the power.*

Thus, in the fifth power the

Exponents of a are	5,	4,	3,	2,	1.	
Exponents of b are		1,	2,	3,	4,	5.

It will be noticed that the sum of the exponents of the letters in any term is equal to the index of the power.

2*d. The coefficient of the first term is one; of the second, the same as the index of the power; and universally, the coefficient of any term, multiplied by the exponent of the leading quantity, and this product, divided by the exponent of the following quantity increased by one, will give the coefficient of the succeeding term.*

Thus, in the fifth power, 5, the coefficient of the second term, multiplied by 4, a's exponent, and divided by 1 plus 1, b's exponent plus 1, $=\frac{5\times 4}{2}=10$, the coefficient of the third term.

The coefficients are repeated in the inverse order after passing the middle term or terms, so that more than half of the coefficients can be written without calculation. The number of terms is always one more than the index of

the power; i. e. the second power has three terms; the third power, four terms; and so on. When the number of terms is even, i. e. when the index of the power is odd, the two central terms have the same coefficient.

3d. *When both terms of the binomial are positive, all the terms of the power are positive; but when the second term is negative, those terms which contain odd powers of the following quantity are negative, and all the others positive; or every alternate term, beginning with the second, is negative, and the others positive.*

1. Expand $(x+y)^8$.

OPERATION.

According to the law, the first term will be x^8,
and the second term $+8x^7y$.

The coefficient of the third term will be $\frac{\overset{4}{\cancel{8}} \times 7}{\cancel{2}}$,
and the third term $+28x^6y^2$.

The coefficient of the fourth term will be $\frac{28 \times \overset{2}{\cancel{6}}}{\cancel{3}}$,
and the fourth term $+56x^5y^3$.

The coefficient of the fifth term will be $\frac{\overset{14}{\cancel{56}} \times 5}{\cancel{4}}$,
and the fifth term $70x^4y^4$.

Having found the preceding coefficients and the coefficient of th middle term, we can write the others at once. Hence,

$$(x+y)^8 = x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8.$$

2. Expand $(a-b)^6$.

Ans. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

3. Expand $(m+n)^7$.

4. Expand $(b-y)^9$.

5. Expand $(a-x)^{10}$.

6. Expand $(b + c)^{11}$.

7. Expand $(x + 1)^5$.

NOTE.— Since all the powers of 1 are 1, 1 is not written when it appears as a factor; but its exponent must be used in obtaining the coefficients.

Ans. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$.

8. Expand $(1 - y)^6$.

Ans. $1 - 6y + 15y^2 - 20y^3 + 15y^4 - 6y^5 + y^6$.

9. Expand $(a - 1)^8$.

130. When the terms of the binomial have coefficients or exponents other than 1, the theorem can be made to apply by treating each term as a single literal quantity. In the expansion, each factor should be enclosed in a parenthesis, and after the expansion of the binomial by the binomial theorem, the work should be completed by the expansion of the enclosed factors, according to the rule for the expansion of monomials.

1. Expand $(2x - y^2)^4$.

OPERATION.

$$(2x)^4 - 4(2x)^3(y^2) + 6(2x)^2(y^2)^2 - 4(2x)(y^2)^3 + (y^2)^4$$

Expanding each factor as indicated, we have

$$16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8$$

2. Expand $(3x^2 - 2y)^5$.

$$(3x^2)^5 - 5(3x^2)^4(2y) + 10(3x^2)^3(2y)^2 - 10(3x^2)^2(2y)^3 + 5(3x^2)(2y)^4 - (2y)^5.$$

Ans. $243x^{10} - 810x^8y + 1080x^6y^2 - 720x^4y^3 + 240x^2y^4 - 32y^5$.

NOTE.— Any letters, as a and b, might be substituted for $3x^2$ and $2y$, and the expansion of $(a - b)^5$ written out, and then the values of a and b substituted.

3. Expand $(a^2 - 3b)^4$.

Ans. $a^8 - 12a^6b + 54a^4b^2 - 108a^2b^3 + 81b^4$.

4. Expand $(x^2 - y^2)^5$.

5. Expand $(2a + 7)^3$.

Ans. $8a^3 + 84a^2 + 294a + 343$.

6. Expand $(2ac - x)^4$.

Ans. $16a^4c^4 - 32a^3c^3x + 24a^2c^2x^2 - 8acx^3 + x^4$.

7. Expand $(a^2x - 2y)^3$.

8. Expand $\left(\frac{1}{2} + x\right)^4$.

Ans. $\frac{1}{16} + \frac{x}{2} + \frac{3x^2}{2} + 2x^3 + x^4$.

9. Expand $\left(\frac{a}{2} - \frac{x}{3}\right)^5$.

Ans. $\frac{a^5}{32} - \frac{5a^4x}{48} + \frac{5a^3x^2}{36} - \frac{5a^2x^3}{54} + \frac{5ax^4}{162} - \frac{x^5}{243}$.

10. Expand $\left(\frac{x}{2} - 1\right)^6$.

11. Expand $\left(\frac{2x}{3} - \frac{3y}{4}\right)^3$.

Ans. $\frac{8x^3}{27} - x^2y + \frac{9xy^2}{8} - \frac{27y^3}{64}$.

12. Expand $\left(\frac{1}{x} + \frac{1}{y}\right)^4$.

Ans. $\frac{1}{x^4} + \frac{4}{x^3y} + \frac{6}{x^2y^2} + \frac{4}{xy^3} + \frac{1}{y^4}$.

13. Expand $\left(ac - \frac{1}{4}\right)^5$.

14. Expand $\left(x + \frac{1}{x}\right)^6$.

15. Expand $\left(1 - \frac{x}{y}\right)^7$.

16. Expand $\left(2a^2 - \frac{1}{a}\right)^5$.

17. Expand $\left(\frac{x}{2} + 1\right)^8$.

18. Expand $\left(\frac{1}{a} - \frac{1}{b}\right)^8$.

131. The Binomial Theorem can be applied to the expansion of a polynomial. Thus, in $a + b - c$, $a + b$ can be treated as a single term, and the quantity can be written $(a + b) - c$. In like manner, $a + b + x - y$ can be written $(a + b) + (x - y)$. In such cases it is easier to substitute a single letter for the enclosed terms, and after the expansion to substitute the proper values.

1. Expand $(a + b - c)^3$.

OPERATION.

Put $\qquad a + b = x$

$$(x - c)^3 = x^3 - 3x^2c + 3xc^2 - c^3$$

Substituting for x, its value, $a + b$;

$$(a + b - c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$$

2. Expand $(2a - b - c - d)^2$.

NOTE.—For $2a - b - c - d$ write $(2a - b) - (c + d)$.

Ans. $4a^2 - 4ab + b^2 - 4ac - 4ad + 2bc + 2bd + c^2 + 2cd + d^2$.

3. Expand $(3x - \frac{1}{2}y - a + b)^2$.

4. Expand $(\frac{1}{4}x - a + b)^3$.

EVOLUTION.

132. EVOLUTION is the process of extracting a root of a quantity. It is the reverse of Involution.

133. A ROOT of any quantity is a quantity which taken as a factor a given number of times will produce the given quantity.

The number of times the root is to be taken as a factor depends upon the name of the root. Thus, the second or square root of a quantity is a quantity which taken twice as a factor will produce the given quantity; the third or cube root is a quantity which taken three times as a factor will produce the given quantity; and so on.

A Root is indicated by the radical sign $\sqrt{\ }$, or by a fractional exponent. Thus,

$\sqrt{x}$, or $x^{\frac{1}{2}}$ indicates the square root of x.
$\sqrt[3]{x}$, or $x^{\frac{1}{3}}$ " " cube " " "
$\sqrt[m]{x}$, or $x^{\frac{1}{m}}$ " " mth " " "

134. A root and a power may be indicated at the same time. Thus, $\sqrt[3]{x^4}$, or $x^{\frac{4}{3}}$, indicates the cube root of the fourth power of x, or the fourth power of the cube root of x; *for a power of a root of a quantity is equal to the same root of the same power of the quantity.* $\sqrt[3]{8^2}$ or $8^{\frac{2}{3}}$ is the square of the cube root of 8, or the cube root of the square of 8, i. e. 4.

135. A *perfect power* is a quantity whose root can be found. A *perfect square* is one whose square root can be found; a *perfect cube* is one whose cube root can be found; and so on.

136. Since Evolution is the reverse of Involution, the rules for Evolution are derived at once from those of Involution. And therefore, as according to Art. 125 an odd power of any quantity has the same sign as the quantity itself, and an even power is always positive, we have for the signs in evolution the following

RULE.

An odd root of a quantity has the same sign as the quantity itself.

An even root of a positive quantity is either positive or negative.

An even root of a negative quantity is impossible, or imaginary.

SQUARE ROOT OF NUMBERS.

137. THE SQUARE ROOT of a number is a number which, taken twice as a factor, will produce the given number.

138. *The square of a number has twice as many figures as the root, or one less than twice as many.* Thus,

Roots,	1,	10,	100,	1000.
Squares,	1,	100,	10000,	1000000.

The square of any number less than 10 must be less than 100; but any number less than 10 is expressed by one figure, and any number less than 100 by less than three figures; i. e. the square of a number consisting of one figure is a number of either one or two figures. The square of any number between 10 and 100 must be between 100 and 10000; i. e. must contain more than two figures and less than five. And the square of any number between 100 and 1000 must contain more than four figures and less than seven.

Hence, to ascertain the number of figures in the square root of a given number,

Beginning at units, point off the number into periods of two figures each; there will be as many figures in the root as there are periods, and for the incomplete period at the left, if any, one more.

139. To extract the square root of a number.

1. Find the square root of 5329.

From the preceding explanation, it is evident that the square root of 5329 is a number of two figures, and that the tens figure of the root is the square root of the greatest perfect square in 53; i. e. $\sqrt{49}$, or 7. Now, if we represent the tens of the root by a and the units by b, $a + b$ will represent the root; and the given number will be

$$(a + b)^2 = a^2 + 2\,a\,b + b^2.$$

Now $\qquad a^2 = 70^2 = 4900;$

therefore, $\qquad 2\,a\,b + b^2 = 5329 - 4900 = 429.$

But $\qquad 2\,a\,b + b^2 = (2\,a + b)\,b;$

If therefore 429 is divided by $2a+b$, it will give b the units of the root. But b is unknown, and is small compared with $2a$; we can therefore use $2a = 140$ as a trial divisor. $429 \div 140$, or $42 \div 14 = 3$, a number that cannot be too small but may be too great, because we have divided by $2a$ instead of $2a+b$. Then $b = 3$, and $2a + b = 140 + 3 = 143$, the true divisor; and $(2a+b)b = 143 \times 3 = 429$; and therefore 3 is the unit figure of the root, and 73 is the required root. The work will appear as follows:—

OPERATION.

$$
\begin{array}{rrl}
 & \dot{5}3\dot{2}9\,(73 & \qquad a = 70 \\
 & 49 & \qquad b = 3 \\
\cline{2-2}
2a+b = 143) & 429 & \\
(2a+b)b = & 429 & \\
\cline{2-2}
\end{array}
$$

Hence, to extract the square root of a number,

RULE.

Separate the given number into periods of two figures each, by placing a dot over units, hundreds, &c.

Find the greatest square in the left-hand period, and place its root at the right.

Subtract the square of this root figure from the left-hand period, and to the remainder annex the next period for a dividend.

Double the root already found for a TRIAL DIVISOR, *and, omitting the right-hand figure of the dividend, divide, and place the quotient as the next figure of the root, and also at the right of the trial divisor for the* TRUE DIVISOR.

Multiply the true divisor by this new root figure, subtract the product from the dividend, and to the remainder annex the next period, for a new dividend.

Double the part of the root already found for a trial divisor, and proceed as before, until all the periods have been employed.

NOTE 1. — When a root figure is 0, annex 0 also to the trial divisor, and bring down the next period to complete the new dividend.

NOTE 2. — If there is a remainder, after using all the periods in the given example, the operation may be continued at pleasure by annexing successive periods of ciphers as decimals.

NOTE 3. — In extracting the root of any number, integral or decimal, place the first point over *unit's* place; and in extracting the square root, over every *second* figure from this. If the last period in the decimal periods is not full, annex 0.

2. Find the square root of 46225.

OPERATION.

$$\begin{array}{r|l} & \dot{4}6\dot{2}2\dot{5}\,(215 \\ & \underline{4} \\ 41) & 62 \\ & \underline{41} \\ 425) & 2125 \\ & \underline{2125} \end{array}$$

We suppose at first that a represents the hundreds of the root, and b the tens; proceeding as in Ex. 1, we have 21 in the root. Then letting a represent the hundreds and tens together, i. e. 21 tens, and b the units, we have $2a$, the 2d trial divisor, $= 42$ tens; and therefore $b = 5$; and $2a + b = 425$; and 215 is the required root.

3. Find the square root of 5013.4.

OPERATION.

$$\begin{array}{r|l} & 5\dot{0}1\dot{3}.4\dot{0}\,(70.805+ \\ & \underline{49} \\ 140.8) & 113.40 \\ & \underline{112.64} \\ 141.605) & .760000 \\ & \underline{.708025} \end{array}$$

4. Find the square root of 288369. Ans. 537.

5. Find the square root of 42849. Ans. 207.

6. Find the square root of 173.261. Ans. 13.16+.

7. Find the square root of .9. Ans. .948+.

8. Find the square root of 2. Ans. 1.4142+.

9. Find the square root of 484.

10. Find the square root of 48.4.

11. Find the square root of .064.

12. Find the square root of .00016.

NOTE. — As a fraction is involved by involving both numerator and denominator (Art. 127), the square root of a fraction *is the square root of the numerator divided by the square root of the denominator.*

13. What is the square root of $\frac{4}{9}$? Ans. $\frac{2}{3}$.

14. What is the square root of $\frac{16}{49}$?

15. What is the square root of $\frac{8}{50}$? $\frac{8}{50} = \frac{4}{25}$. Ans. $\frac{2}{5}$.

NOTE. — If both terms of the fraction are not perfect squares, and cannot be made so, reduce the fraction to a decimal, and then find the square root of the decimal. A mixed number must be reduced to an improper fraction, or the fractional part to a decimal, before its root can be found.

16. What is the square root of $\frac{2}{7}$? Ans. .53+.

17. What is the square root of $\frac{2}{217}$?

18. What is the square root of $\frac{4}{11}$?

19. What is the square root of $7\frac{5}{8}$?

CUBE ROOT OF NUMBERS.

140. The CUBE ROOT of a number is a number which, taken three times as a factor, will produce the given number.

141. *The cube of a number consists of three times as many figures as the root, or of one or two less than three times as many.*

Roots,	1,	10,	100,	1000.
Cubes,	1,	1000,	1000000,	1000000000.

The cube of any number less than 10 must be less than 1000; but any number less than 10 is expressed by one figure, and any

number less than 1000 by less than four figures; i. e. the cube of a number consisting of one figure is a number of less than four figures. The cube of any number between 10 and 100 must be between 1000 and 1000000; i. e. must contain more than three figures and less than seven. And in the same way we see that the cube of any number between 100 and 1000 must contain more than six figures and less than ten.

Hence, to ascertain the number of figures in the cube root of a given number,

Beginning at units, point off the number into periods of three figures each; there will be as many figures in the root as there are periods, and for the incomplete period at the left, if any, one more.

142. To extract the cube root of a number.

1. Find the cube root of 42875.

From the preceding explanation, it is evident that the cube root of 42875 is a number of two figures, and that the tens figure of the root is the cube root of the greatest perfect cube in 42; i. e. $\sqrt[3]{27}$, or 3. Now, if we represent the tens of the root by a and the units by b, $a + b$ will represent the root, and the given number will be

$$(a + b)^3 = a^3 + 3\,a^2 b + 3\,a\,b^2 + b^3.$$

Now $$a^3 = 30^3 = 27000;$$

therefore, $$3\,a^2 b + 3\,a\,b^2 + b^3 = 42875 - 27000 = 15875.$$

But $$3\,a^2 b + 3\,a\,b^2 + b^3 = (3\,a^2 + 3\,a\,b + b^2)\,b.$$

If therefore 15875 is divided by $3\,a^2 + 3\,a\,b + b^2$ it will give b, the units of the root. But b, and therefore $3\,a\,b + b^2$, a part of the divisor, is unknown, and we must use $3\,a^2 = 2700$ as a trial divisor. $15875 \div 2700$, or $158 \div 27 = 5$, a number that cannot be too small but may be too great, because we have divided by $3\,a^2$ instead of the true divisor, $3\,a^2 + 3\,a\,b + b^2$. Then $b = 5$, and $3\,a^2 + 3\,a\,b + b^2 = 2700 + 450 + 25 = 3175$, the true divisor; and $(3\,a^2 + 3\,a\,b + b^2)\,b = 3175 \times 5 = 15875$, and therefore 5 is the unit's figure of the root, and 35 is the required root. The work will appear as follows:—

OPERATION.

$$\begin{array}{lr|l}
 & & 4\dot{2}87\dot{5}\,(35 \text{ Root.} \\
 & & 27 \\
\cline{3-3}
\text{Trial divisor,} & 3a^2 = 2700 & 15875 \text{ Dividend.} \\
 & 3ab = 450 & \\
 & b^2 = 25 & \\
\cline{2-2}
\text{True divisor, } 3a^2 + 3ab + b^2 = & 3175 & 15875 \\
\cline{3-3}
\end{array}$$

Hence, to extract the cube root of a number,

RULE.

Separate the number into periods of three figures each, by placing a dot over units, thousands, &c.

Find the greatest cube in the left-hand period, and place its root at the right.

Subtract this cube from the left-hand period, and to the remainder annex the next period for a dividend.

Square the root figure, annex two ciphers, and multiply this result by three for a TRIAL DIVISOR; *divide the dividend by the trial divisor, and place the quotient as the next figure of the root.*

Multiply this root figure by the part of the root previously obtained, annex one cipher and multiply this result by three; add the last product and the square of the last root figure to the trial divisor, and the SUM *will be the* TRUE DIVISOR.

Multiply the true divisor by the last root figure, subtract the product from the dividend, and to the remainder annex the next period for a dividend.

Find a new trial divisor, and proceed as before, until all the periods have been employed.

NOTE 1.—The notes under the rule in square root (Art. 139) apply also to the extraction of the cube root, except that 00 must be annexed to the trial divisor when the root figure is 0, and after placing the first point over units the point must be placed over every *third* figure from this.

NOTE 2.—As the trial divisor may be much less than the true

divisor, the quotient is frequently too great, and a less number must be placed in the root.

2. Find the cube root of 18191447.

OPERATION.

			$\dot{1}8\,1\,9\,\dot{1}\,4\,4\,\dot{7}$ (263	
			8	
1st Trial Divisor,	$3a^2$	= 1200	10191	1st Dividend.
	$3ab$	= 360		
	b^2	= 36		
1st True Divisor,	$3a^2+3ab+b^2$	= 1596	9576	
2d Trial Divisor,	$3a^2$	= 202800	615447	2d Div.
	$3ab$	= 2340		
	b^2	= 9		
2d True Divisor,	$3a^2+3ab+b^2$	= 205149	615447	

We suppose at first that a represents the hundreds of the root and b the tens: proceeding as in Ex. 1, we have 26 in the root. Then letting a represent the hundreds and tens together, i. e. 26 tens, and b the units, we have $3a^2$, the 2d trial divisor, $= 202800$; and therefore $b = 3$; and $3a^2 + 3ab + b^2$, the 2d true divisor, $= 205149$; and 263 is the required root.

NOTE. — Though the 1st trial divisor is contained more than 8 times in the dividend, yet the root figure is only 6.

3. Find the cube root of 68716.47.

OPERATION.

	$6\dot{8}\,7\,1\,\dot{6}.4\,7\,\dot{0}$ (40.95+
	64
4800.00	4716.470
108.00	
.81	
4908.81	4417.929
5018.4300	298.541000
6.1350	
.0025	
5024.5675	251.228375
	47.312625

4. Find the cube root of 2924207. Ans. 143.

5. Find the cube root of 8120601. Ans. 201.

6. Find the cube root of 36926037.

7. Find the cube root of 67917.312.

8. Find the cube root of 46417.8.

9. Find the cube root of .8. Ans. .928+.

10. Find the cube root of .17164.

11. Find the cube root of .0064.

12. Find the cube root of 25.00017.

13. Find the cube root of 2.7.

NOTE.—As a fraction is involved by involving both numerator and denominator (Art. 127), the cube root of a fraction is *the cube root of the numerator divided by the cube root of the denominator.*

14. What is the cube root $\frac{8}{27}$? Ans. $\frac{2}{3}$.

15. What is the cube root of $\frac{64}{729}$?

16. What is the cube root of $\frac{250}{432}$? $\frac{250}{432} = \frac{125}{216}$.
Ans. $\frac{5}{6}$.

NOTE.—If both terms of the fraction are not perfect cubes, and cannot be made so, reduce the fraction to a decimal, and then find the cube root of the decimal. A mixed number must be reduced to an improper fraction, or the fractional part to a decimal, before its root can be found.

17. What is the cube root of $\frac{8}{11}$? Ans. .899+.

18. What is the cube root of $\frac{7}{81}$?

19. What is the cube root of $3\frac{1}{4}$?

20. What is the cube root of $117\frac{6}{7}$?

EVOLUTION OF MONOMIALS.

143. As Evolution is the reverse of Involution, and since to involve a monomial (Art. 126) we multiply the exponent of each letter by the index of the required power, and prefix the required power of the numerical coefficient,

Hence, to find the root of a monomial,

RULE.

Divide the exponent of each letter by the index of the required root, and prefix the required root of the numerical coefficient.

NOTE 1.—The rule for the signs is given in Art. 136. As an even root of a positive quantity may be either positive or negative, we prefix to such a root the sign $\pm$; read, plus or minus.

NOTE 2.—It follows from this rule that *the root of the product of several factors is equal to the product of the roots.* Thus, $\sqrt{36} = \sqrt{4}\,\sqrt{9} = 6$.

1. Find the cube root of $8\,x^3\,y^6$. Ans. $2\,x\,y^2$.
2. Find the square root of $4\,x^2$. Ans. $\pm\,2\,x$.
3. Find the third root of $-\,125\,a^6\,x$. Ans. $-\,5\,a^2\,x^{\frac{1}{3}}$.
4. Find the fourth root of $81\,a^{-4}\,b$. Ans. $\pm\,3\,a^{-1}\,b^{\frac{1}{4}}$.
5. Find the fifth root of $32\,a^{10}\,b^3$. Ans. $2\,a^2\,b^{\frac{3}{5}}$.
6. Find the cube root of $-\,729\,x^3\,y^9$. Ans. $-\,9\,x\,y^3$.
7. Find the fourth root of $256\,x^4\,y^2$.
8. Find the cube root of $-\,512\,a^{-3}\,b^3$.
9. Find the fifth root of $243\,x^5\,y^{10}$.

NOTE. — As a fraction is involved by involving both numerator and denominator (Art. 127), a fraction must be evolved by evolving both numerator and denominator.

10. Find the square root of $\frac{4a^2}{9y^4}$. Ans. $\pm\frac{2a}{3y^2}$.

Perform the operations indicated in the following expressions: —

11. $\sqrt[3]{-729a^3b^6c^9}$.

12. $(49a^2x^4y^6)^{\frac{1}{2}}$.

13. $\sqrt{\frac{9a^2x^4}{36b^4y^6}}$.

14. $\sqrt[m]{a^m x^{mn}}$.

15. $(256a^4x^{10}y^{16})^{\frac{1}{4}}$.

16. $\sqrt[4]{81a^2b^8}$.

17. $\sqrt[m]{a^m b^{2m} c^{mn}}$.

SQUARE ROOT OF POLYNOMIALS.

144. In order to discover a method for extracting the square root of a polynomial, we will consider the relation of $a+b$ to its square, $a^2+2ab+b^2$. The first term of the square contains the square of the first term of the root; therefore the square root of the first term of the square will be the first term of the root. The second term of the square contains twice the product of the two terms of the root; therefore, if the second term of the square, $2ab$, is divided by twice the first term of the root, $2a$, we shall have the second term of the root b. Now, $2ab+b^2=(2a+b)b$; therefore, if to the trial divisor $2a$ we add b, when it has been found, and then

multiply the corrected divisor by b, the product will be equal to the remaining terms of the power after a^2 has been subtracted.

The process will appear as follows:—

OPERATION.

$$
\begin{array}{rl}
 & a^2 + 2ab + b^2 \,(a + b \\
 & \underline{a^2 \qquad\qquad\quad} \\
2a + b) & 2ab + b^2 \\
 & \underline{2ab + b^2}
\end{array}
$$

Having written a, the square root of a^2, in the root, we subtract its square (a^2) from the given polynomial, and have $2ab + b^2$ left. Dividing the first term of this remainder, $2ab$, by $2a$, which is double the term of the root already found, we obtain b, the second term of the root, which we add both to the root and to the divisor. If the product of this corrected divisor and the last term of the root is subtracted from $2ab + b^2$, nothing remains.

145. Since a polynomial can always be written and involved like a binomial, as shown in Art. 131, we can apply the process explained in the preceding Article to finding the root, when this root consists of any number of terms.

1. Find the square root of $a^2 + 2ab + b^2 - 2ac - 2bc + c^2$.

OPERATION.

$$
\begin{array}{rl}
 & a^2 + 2ab + b^2 - 2ac - 2bc + c^2 \,(a + b - c \\
 & \underline{a^2 \qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
2a + b) & 2ab + b^2 \\
 & \underline{2ab + b^2 \qquad\qquad\qquad\qquad} \\
2a + 2b - c) & -2ac - 2bc + c^2 \\
 & \underline{-2ac - 2bc + c^2}
\end{array}
$$

Proceeding as before, we find the first two terms of the root $a + b$. Considering $a + b$ as a single quantity, we divide the remainder $-2ac - 2bc + c^2$ by twice this root, and obtain $-c$, which we write both in the root and in the divisor. If this corrected divisor is multiplied by $-c$, and the product subtracted from the dividend, nothing remains.

Hence, to extract the square root of a polynomial,

RULE.

Arrange the terms according to the powers of some letter.

Find the square root of the first term, and write it as the first term of the root, and subtract its square from the given polynomial.

Divide the remainder by double the root already found, and annex the result both to the root and to the divisor.

Multiply the corrected divisor by this last term of the root, and subtract the product from the last remainder. Proceed as before with the remainder, if there is any.

2. Find the square root of $4x^2 - 4xy^2 + y^4$.
Ans. $2x - y^2$.

3. Find the square root of $a^2 + 2ab + b^2 + 4ac + 4bc + 4c^2$.
Ans. $a + b + 2c$.

4. Find the square root of $9x^4 - 12x^3 + 4x^2 + 6ax^2 - 4ax + a^2$.
Ans. $3x^2 - 2x + a$.

5. Find the square root of $4a^2 + 8ab - 4a + 4b^2 - 4b + 1$
Ans. $2a + 2b - 1$.

6. Find the square root of $25x^4 - 10x^3 + 6x^2 - x + \frac{1}{4}$.
Ans. $5x^2 - x + \frac{1}{2}$.

7. Find the square root of $x^6 + 2x^5 - x^4 - 2x^3 + x^2$.

8. Find the square root of $4a^2 - 4ab + b^2 - 4ac - 4ad + 2bc + 2bd + c^2 + 2cd + d^2$.
Ans. $2a - b - c - d$.

9. Find the square root of $x^6 - 4x^5 + 6x^4 - 6x^3 + 5x^2 - 2x + 1$.

10. Find the square root of $4a^4 + 8a^3b - 8a^2b^2 - 12ab^3 + 9b^4$.

NOTE 1. — According to the principles of Art. 136, the signs of the answers given above may all be changed, and still be correct.

NOTE 2. — *No binomial can be a perfect square.* For the square of a monomial is a monomial, and the square of the polynomial with the least number of terms, that is, of a binomial, is a trinomial.

NOTE 3. — A trinomial is a perfect square when two of its terms are perfect squares and the remaining term is equal to twice the product of their square roots. For,

$$(a + b)^2 = a^2 + 2\,a\,b + b^2$$
$$(a - b)^2 = a^2 - 2\,a\,b + b^2$$

Therefore the square root of $a^2 \pm 2\,a\,b + b^2$ is $a \pm b$. Hence, to obtain the square root of a trinomial which is a perfect square,

Omitting the term that is equal to twice the product of the square roots of the other two, connect the square roots of the other two by the sign of the term omitted.

11. Find the square root of $\frac{x^2}{4} - \frac{x\,y}{2} + \frac{y^2}{4}$

Ans. $\frac{x}{2} - \frac{y}{2}$.

12. Find the square root of $x^2 + 2\,x + 1$.

Ans. $x + 1$.

13. Find the square root of $4\,x^2 - 8\,x\,y + 4\,y^2$.

14. Find the square root of $\frac{a^2}{9} - 2\,a\,b + 9\,b^2$.

15. Find the square root of $16\,y^2 + 40\,y\,z^2 + 25\,z^4$.

NOTE. — By the rule for extracting the square root, any root whose index is any power of 2 can be obtained by successive extractions of the square root. Thus, the fourth root is the square root of the square root; the eighth root is the square root of the square root of the square root; and so on.

16. Find the fourth root of $a^8 - 12\,a^6\,b + 54\,a^4\,b^2 - 108\,a^2\,b^3 + 81\,b^4$. Ans. $a^2 - 3\,b$.

17. Find the fourth root of $\frac{1}{x^4}+\frac{4}{x^3y}+\frac{6}{x^2y^2}+\frac{4}{xy^3}+\frac{1}{y^4}$.

Ans. $\frac{1}{x}+\frac{1}{y}$.

18. Find the fourth root of $x^8-4x^7+10x^6-16x^5+19x^4-16x^3+10x^2-4x+1$. Ans. x^2-x+1.

146. To find any root of a polynomial.

Since, according to the Binomial Theorem, when the terms of a power are arranged according to the power of some letter beginning with its highest power, the first term contains the first term of the root raised to the given power, therefore, if we take the required root of the first term, we shall have the first term of the root. And since the second term of the power contains the second term of the root multiplied by the next inferior power of the first term of the root with a coefficient equal to the index of the root, therefore if we divide the second term of the power by the first term of the root raised to the next inferior power with a coefficient equal to the index of the root, we shall have the second term of the root. In accordance with these principles, to find any root of a polynomial we have the following

RULE.

Arrange the terms according to the powers of some letter.

Find the required root of the first term, and write it as the first term of the root.

Divide the second term of the polynomial by the first term of the root raised to the next inferior power and multiplied by the index of the root.

Involve the whole of the root thus found to the given power, and subtract it from the polynomial.

If there is any remainder, divide its first term by the divisor first found, and the quotient will be the third term of the root.

Proceed in this manner till the power obtained by involving the root is equal to the given polynomial.

NOTE 1. — This rule verifies itself. For the root, whenever a new term is added to it, is involved to the given power, and whenever the root thus involved is equal to the given polynomial, it is evident that the required root is found.

NOTE 2. — As *powers* and *roots* are correlative words, we have used the phrase *given power*, meaning the power whose index is equal to the index of the required root, and the phrase *next inferior power* meaning that power whose index is one less than the index of the required root.

1. Find the cube root of $a^6 - 3a^5 + 5a^3 - 3a - 1$.

OPERATION.

$$\begin{array}{rl}
\text{Constant divisor, } 3a^4) & a^6 - 3a^5 + 5a^3 - 3a - 1\ (a^2 - a - 1 \\
 & \underline{a^6 - 3a^5 + 3a^4 - a^3} \\
 & \underline{\qquad\qquad -3a^4}, \text{ 1st term of remainder.} \\
 & a^6 - 3a^5 + 5a^3 - 3a - 1
\end{array}$$

The first term of the root is a^2, the cube root of a^6. a^2 raised to the next inferior power, i. e. to the second power, with the coefficient 3, the index of the root, gives $3a^4$, which is the constant divisor. $-3a^5$, the second term of the polynomial, divided by $3a^4$, gives $-a$, the second term of the root. $(a^2 - a)^3 = a^6 - 3a^5 + 3a^4 - a^3$; and subtracting this from the polynomial, we have $-3a^4$ as the first term of the remainder. $-3a^4$ divided by $3a^4$ gives -1, the third term of the root. $(a^2 - a - 1)^3 =$ the given polynomial, and therefore the correct root has been found.

2. Find the fourth root of $16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8$.

OPERATION.

$$\begin{array}{rl}
4 \times (2x)^3 = 32x^3) & 16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8\ (2x - y^2 \\
 & \underline{16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8}
\end{array}$$

3. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$.

4. Find the fourth root of $16a^4c^4 - 32a^3c^3x + 24a^2c^2x^2 - 8acx^3 + x^4$.

7*

SECTION XVII.

RADICALS.

147. A RADICAL is the indicated root of any quantity, as $\sqrt{x}$, $a^{\frac{1}{3}}$, $\sqrt{2}$, $3^{\frac{1}{2}}$, &c.

148. In distinction from *radicals*, other quantities are called *rational* quantities.

149. The factor standing before the radical is the *coefficient* of the radical. Thus, 2 is the coefficient of $\sqrt{2}$ in the expression $2\sqrt{2}$.

150. SIMILAR RADICALS are those which have the same quantity under the same radical sign. Thus, $\sqrt{a}$, $2\sqrt{a}$, and $x\sqrt{a}$ are *similar* radicals; but $2\sqrt{a}$ and $2\sqrt{b}$, or $2x^{\frac{1}{3}}$ and $2x^{\frac{1}{5}}$ are *dissimilar* radicals.

151. A SURD is a quantity whose indicated root cannot be found. Thus, $\sqrt{2}$ is a surd.

The various operations in radicals are presented under the following cases.

CASE I.

152. To reduce a radical to its simplest form.

NOTE. — A radical is in its simplest form when it contains no factor whose indicated root can be found.

1. Reduce $\sqrt{75a^2b}$ to its simplest form.

OPERATION.

$$\sqrt{75a^2b} = \sqrt{25a^2 \times 3b} = \sqrt{25a^2} \times \sqrt{3b} = 5a\sqrt{3b}$$

We first resolve $75a^2b$ into two factors, one of which, $25a^2$, is the greatest perfect square which it contains; then, as the root of

the product is equal to the product of the roots (Art. 143, Note 2), we extract the square root of the perfect square $25a^2$, and annex to this root the factor remaining under the radical. Hence,

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect power of the same name as the root. Extract the root of the perfect power, multiply it by the coefficient of the radical, if it has any, and annex to the result the other factor, with the radical sign between them.

Reduce the following expressions to their simplest form: —

2. $\sqrt{12x}$. Ans. $2\sqrt{3x}$.

3. $\sqrt{49x^7}$. Ans. $7x^3\sqrt{x}$.

4. $\sqrt[3]{72a^3b^2}$. Ans. $2a\sqrt[3]{9b^2}$.

5. $5\sqrt[4]{64ab^4}$. Ans. $10b\sqrt[4]{4a}$.

6. $3\sqrt{147a^2b^4}$. Ans. $21ab^2\sqrt{3}$.

7. $25\sqrt[3]{56x}$. Ans. $50\sqrt[3]{7x}$.

8: $4\sqrt{128x^2y}$.

9. $\sqrt[3]{343x^8}$.

10. $\sqrt{\frac{27a^2c}{128x^6y}}$.

$$\sqrt{\frac{27a^2c}{128x^6y}} = \sqrt{\frac{9a^2}{64x^6}}\sqrt{\frac{3c}{2y}} = \frac{3a}{8x^3}\sqrt{\frac{3c}{2y}}, \text{ Ans.}$$

11. $\sqrt[3]{\frac{8x^6y^2}{54z}}$. Ans. $\frac{2x^2}{3}\sqrt[3]{\frac{y^2}{2z}}$.

12. $\sqrt{16x^2y^2 - 32x^4y^6}$.

$$\sqrt{16x^2y^2 - 32x^4y^6} = \sqrt{16x^2y^2}\sqrt{1 - 2x^2y^4}$$
$$= 4xy\sqrt{1 - 2x^2y^4}, \text{ Ans.}$$

13. $4 \sqrt[3]{81 a^3 c + 27 a^3}$. Ans. $12 a \sqrt[3]{3 c + 1}$.

14. $(a + b) \sqrt{3 a^2 - 6 a b + 3 b^2}$. Ans. $(a^2 - b^2) \sqrt{3}$.

15. $7 \sqrt[3]{250 x^6 y^9 - 125 x^3 y^3}$.

16. $(x - y)\,(a^3 x - a^6 y)^{\frac{1}{3}}$.

17. $(a^3 + a^3 b^2)^{\frac{1}{3}}$.

18. $\sqrt{-16}$.

$$\sqrt{-16} = \sqrt{16}\,\sqrt{-1} = 4\sqrt{-1}, \text{ Ans.}$$

19. $\sqrt[4]{-1250}$.

20. $\sqrt{19 a^2 - 4 b^2}$.

153. When a fraction is under the radical sign, it can be transformed so as to have only an integral quantity under the radical sign, *by multiplying both terms of the fraction by that quantity which will make its denominator a perfect power of the same name as the root, and then removing a factor according to the Rule in Art.* 152.

1. Reduce $\sqrt[3]{\frac{2}{9}}$ to its simplest form.

OPERATION.

$$\sqrt[3]{\frac{2}{9}} = \sqrt[3]{\frac{6}{27}} = \sqrt[3]{\frac{1}{27}}\,\sqrt[3]{6} = \frac{1}{3}\sqrt[3]{6}$$

Transform each of the following expressions so as to have only an integral quantity under the radical sign.

2. $\frac{1}{2}\sqrt{\frac{2}{3}}$. Ans. $\frac{1}{6}\sqrt{6}$.

3. $4\sqrt[3]{\frac{x^3}{3}}$. Ans. $\frac{4 x}{3}\sqrt[3]{9}$.

4. $\frac{2}{5}\sqrt[4]{\frac{1}{7}}$. Ans. $\frac{2}{35}\sqrt[4]{343}$.

5. $\frac{a}{b}\sqrt{\frac{17}{18 a}}$. Ans. $\frac{1}{6 b}\sqrt{34 a}$.

6. $\sqrt{\frac{27x}{50}}$. Ans. $\frac{3}{10}\sqrt{6x}$.

7. $2\sqrt{\frac{7}{8x}}$. Ans. $\frac{1}{2x}\sqrt{14x}$.

8. $14\left(\frac{54}{147}\right)^{\frac{1}{2}}$. $14\left(\frac{54}{147}\right)^{\frac{1}{2}} = 14\left(\frac{9}{49}\right)^{\frac{1}{2}}\left(\frac{6}{3}\right)^{\frac{1}{2}}$. Ans. $6\sqrt{2}$.

9. $\sqrt{\frac{294}{845}}$. Ans. $\frac{7}{65}\sqrt{30}$.

10. $\sqrt[3]{\frac{432}{1875}}$.

11. $(a+b)\sqrt{\frac{a-b}{a+b}}$. Ans. $\sqrt{a^2-b^2}$.

CASE II.

154. To reduce a rational quantity to the form of a radical.

1. Reduce $3x^2$ to the form of the cube root.

OPERATION.

$$3x^2 = \sqrt[3]{27x^6}$$

Since $3x^2$ is to be placed under the form of the cube root without changing its value, we cube it and then place the radical sign, $\sqrt[3]{}$, over it. It is evident that $\sqrt[3]{27x^6} = 3x^2$. Hence,

RULE.

Involve the quantity to the power denoted by the index of the root required, and place the corresponding radical sign over the power thus produced.

2. Reduce $4a^3b$ to the form of the square root.

Ans. $\sqrt{16a^6b^2}$.

3. Reduce $2ab^2c^{-1}$ to the form of the fifth root.

Ans. $\sqrt[5]{32a^5b^{10}c^{-5}}$.

4. Reduce $\frac{1}{3}a^{\frac{1}{3}}c^2$ to the form of the cube root.

5. Reduce $\frac{2a^3b}{3xy^{\frac{1}{4}}}$ to the form of the fourth root.

6. Reduce $x - 2y$ to the form of the square root.

Ans. $\sqrt{x^2 - 4xy + 4y^2}$.

155. On the same principle the rational coefficient of a radical can be placed under the radical sign, *by involving the coefficient to a power of the same name as the root indicated by the radical sign, multiplying it by the radical quantity, and placing the given radical sign over the product.*

1. Place the coefficient of $5\sqrt[3]{2y}$ under the radical sign.

OPERATION.

$$5\sqrt[3]{2y} = \sqrt[3]{125}\sqrt[3]{2y} = \sqrt[3]{250y}$$

In the following examples, place the coefficient under the radical sign.

2. $3\sqrt[4]{4x^3y}$. Ans. $\sqrt[4]{324x^3y}$.

3. $2xy\sqrt[3]{2x^2y}$. Ans. $\sqrt[3]{16x^5y^4}$.

4. $\frac{3}{4}\sqrt{\frac{1}{5}}$.

5. $\frac{1}{2}\sqrt{14}$.

6. $(a - b)\sqrt[3]{\frac{a}{a-b}}$. Ans. $\sqrt[3]{a^3 - 2a^2b + ab^2}$.

7. $4xy\sqrt{1 - 2x^2y^4}$.

CASE III.

156. To reduce radicals having different indices to equivalent ones having a common index.

1. Reduce $\sqrt{a}$ and $\sqrt[3]{b}$ to equivalent radicals having a common index.

OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$$
$$b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}$$

In this case we write the radicals with their fractional indices; and then, as the denominator is the index of the root, in order that the two radicals may have the same root-index, we reduce the fractional indices to equivalent ones having a common denominator. It is evident that we have not changed the values of the given radicals by the process. Hence,

RULE.

Reduce the fractional indices to equivalent ones having a common denominator; involve each quantity to the power denoted by the numerator of the reduced index, and indicate the root denoted by the denominator.

2. Reduce $\sqrt[3]{2}$ and $\sqrt[4]{3}$ to equivalent radicals having a common index.

$$\left.\begin{aligned} 2^{\frac{1}{3}} &= 2^{\frac{4}{12}} = \sqrt[12]{2^4} = \sqrt[12]{16} \\ 3^{\frac{1}{4}} &= 3^{\frac{3}{12}} = \sqrt[12]{3^3} = \sqrt[12]{27} \end{aligned}\right\} \text{Ans.}$$

3. Reduce $\sqrt{\frac{1}{5}}$ and $\sqrt[3]{\frac{1}{6}}$ to equivalent radicals having a common index. Ans. $\sqrt[6]{\frac{1}{125}}$ and $\sqrt[6]{\frac{1}{36}}$.

4. Reduce $\sqrt{\frac{x}{2}}$ and $\sqrt[4]{\frac{y}{3}}$ to equivalent radicals having a common index.

5. Reduce $\sqrt[4]{a}$, $\sqrt[3]{a-b}$, and $\sqrt{a+b}$ to equivalent radicals having a common index.

Ans. $\sqrt[12]{a^3}$, $\sqrt[12]{(a-b)^4}$, and $\sqrt[12]{(a+b)^6}$.

6. Reduce $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{3}$ to equivalent radicals having a common index.

7. Reduce $\sqrt[n]{x}$ and $\sqrt[m]{y}$ to equivalent radicals having a common index. Ans. $\sqrt[mn]{x^m}$ and $\sqrt[mn]{y^n}$.

CASE IV.

157. To add radical quantities.

1. Add $\sqrt{x}$ and $\sqrt{y}$. Ans. $\sqrt{x}+\sqrt{y}$.

It is evident that the addition can only be expressed.

2. Add $3\sqrt{x}$ and $5\sqrt{x}$. Ans. $8\sqrt{x}$.

It is evident that 3 times the $\sqrt{x}$ and 5 times the $\sqrt{x}$ make 8 times the $\sqrt{x}$.

3. Add $\sqrt{8}$ and $\sqrt{50}$ together.

OPERATION.

$$\begin{aligned}\sqrt{8} &= 2\sqrt{2}\\ \sqrt{50} &= 5\sqrt{2}\\ \hline \text{Sum} &= 7\sqrt{2}\end{aligned}$$

In this case we make the radical parts similar by reducing them to their simplest form (Art. 152), and then add their coefficients as in Example 2. Hence,

RULE.

Make the radical parts similar when they are not, and prefix the sum of the coefficients to the common radical. If the radical parts are not and cannot be made similar, connect the quantities with their proper signs.

4. Add $2\sqrt{50\,a\,x}$ and $3\sqrt{98\,a\,x}$. Ans. $31\sqrt{2\,a\,x}$.

5. Add $4\sqrt[3]{24\,x^3}$ and $x\sqrt[3]{81}$. Ans. $11\,x\sqrt[3]{3}$.

6. Add $\sqrt{27}$ and $\sqrt{363}$. Ans. $14\sqrt{3}$.

7. Add $\sqrt[4]{512\,x^4}$ and $\sqrt[4]{162\,y^4}$.

Ans. $(4\,x+3\,y)\sqrt[4]{2}$.

8. Add $\sqrt{5}$ and $\sqrt{\frac{1}{5}}$.

$\sqrt{\frac{1}{5}}=\sqrt{\frac{1}{25}}\sqrt{5}=\frac{1}{5}\sqrt{5}$; $\sqrt{5}+\frac{1}{5}\sqrt{5}=\frac{6}{5}\sqrt{5}$, Ans.

9. Add $\sqrt[3]{\frac{3\,x^3}{2}}$ and $\sqrt[3]{\frac{4\,x^3}{9}}$. Ans. $\frac{5\,x}{6}\sqrt[3]{12}$.

10. Add $\sqrt{\frac{5}{9}}$, $10\sqrt{\frac{1}{45}}$, and $6\sqrt{20}$. Ans. $13\sqrt{5}$.

11. Add $\sqrt{10}$ and $\sqrt{20}$.

CASE V.

158. To subtract one radical from another.

1. From $\sqrt{75}$ take $\sqrt{27}$.

OPERATION.

$$\sqrt{75} = 5\sqrt{3}$$
$$\sqrt{27} = 3\sqrt{3}$$
$$\overline{\qquad 2\sqrt{3}}$$

We make the radical parts similar by reducing them to their simplest form (Art. 152). And $3\sqrt{3}$ taken from $5\sqrt{3}$ evidently leaves $2\sqrt{3}$. Hence,

RULE.

Make the radical parts similar when they are not, subtract the coefficient of the subtrahend from that of the minuend, and prefix the difference to the common radical. If the radical parts are not and cannot be made similar, indicate the subtraction by connecting them with the proper sign.

2. From $\sqrt[3]{81}$ take $\sqrt[3]{3}$. Ans. $2\sqrt[3]{3}$.

3. From $9\sqrt{a^2xy^2}$ take $3a\sqrt{xy^2}$. Ans. $6ay\sqrt{x}$.

4. From $7\sqrt{20x}$ take $4\sqrt{45x}$. Ans. $2\sqrt{5x}$.

5. From $\sqrt[3]{500}$ take $\sqrt[3]{108}$. Ans. $2\sqrt[3]{4}$.

6. From $2\sqrt{\frac{5}{16}}$ take $\sqrt{\frac{20}{49}}$. Ans. $\frac{3}{14}\sqrt{5}$.

7. From $\sqrt{\frac{2}{5}}$ take $\sqrt{\frac{5}{32}}$. Ans. $\frac{3}{40}\sqrt{10}$.

8. From $2\sqrt[4]{176x^5}$ take $\sqrt[4]{891x^5}$.

9. From $a\sqrt[5]{x^2}$ take $7\sqrt[5]{a^5x^2}$.

10. From $\sqrt[6]{1174}$ take $\sqrt[6]{1892}$.

CASE VI.

159. To multiply radicals.

1. Multiply $3\sqrt{a}$ by $5\sqrt{b}$.

OPERATION.

$$3\sqrt{a} \times 5\sqrt{b} = 3 \times 5 \times \sqrt{a} \times \sqrt{b} = 15\sqrt{ab}$$

As it makes no difference in what order the factors are taken, we unite in one product the numerical coefficients; and $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (Art. 143, Note 2).

2. Multiply $4\sqrt{2ab}$ by $5\sqrt[3]{3ay}$.

OPERATION.

$$4\sqrt{2ab} = 4\sqrt[6]{8a^3b^3}$$
$$5\sqrt[3]{3ax} = 5\sqrt[6]{9a^2x^2}$$
$$\text{Product} = 20\sqrt[6]{72a^5b^3x^2}$$

We reduce the radical parts to equivalent radicals having a common index (Art. 156), and then multiply as in the preceding example.

3. Multiply $\sqrt[3]{a}$ by $\sqrt{a}$.

OPERATION.

$$a^{\frac{1}{3}} \times a^{\frac{1}{2}} = \sqrt[6]{a^2 \times a^3} = \sqrt[6]{a^5}, \text{ or } a^{\frac{5}{6}}$$

Multiplying as in the preceding examples, we have $\sqrt[6]{a^5}$, or $a^{\frac{5}{6}}$; but $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$; i. e. the index of the product is the sum of the indices of the factors.

From these examples we deduce the following

RULE.

I. *Reduce the radical parts, if necessary, to equivalent radicals having a common index, and to the product of the radical parts placed under the common radical sign prefix the product of their coefficients.*

II. *Roots of the same quantity are multiplied together by adding their fractional indices.*

4. Multiply $3\sqrt{10}$ by $4\sqrt{5}$. Ans. $60\sqrt{2}$.

5. Multiply $4\sqrt[3]{a x^2}$ by $a\sqrt{x}$. Ans. $4 a x \sqrt[6]{a^2 x}$.

6. Multiply $a\sqrt{c}$ by $b\sqrt{c}$. Ans. $a b c$.

7. Multiply $\sqrt{x}$ by $\sqrt[3]{x}$. Ans. $x^{\frac{5}{6}}$.

8. Multiply $\sqrt[3]{7}$ by $\sqrt{7}$. Ans. $7^{\frac{5}{6}}$, or $\sqrt[6]{16807}$.

9. Multiply $\sqrt[m]{x}$ by $\sqrt[n]{x}$. Ans. $\sqrt[mn]{x^{m+n}}$.

10. Multiply $2\sqrt{a+b}$ by $6x\sqrt[3]{a+b}$.
Ans. $12x\sqrt[6]{(a+b)^5}$.

11. Multiply $\sqrt{x}$, $\sqrt[3]{x}$, and $\sqrt[4]{x}$ together. Ans. $x^{\frac{13}{12}}$.

12. Multiply $3\sqrt[3]{\frac{1}{16}}$ by $2\sqrt{8}$. Ans. $6\sqrt[6]{2}$.

13. Multiply $a\sqrt{x^{-1}y}$ by $b\sqrt{xy}$. Ans. $a b y$.

14. Multiply $(a+b)^{\frac{1}{2}}$ by $(a-b)^{\frac{1}{2}}$. Ans. $(a^2-b^2)^{\frac{1}{2}}$.

15. Multiply $\frac{1}{3}\sqrt{\frac{1}{7}}$ by $3\sqrt{7}$.

16. Multiply $2\sqrt{5}$ by $4\sqrt{8}$.

CASE VII.

160. To divide radicals.

1. Divide $60\sqrt{15x}$ by $4\sqrt{5x}$.

OPERATION.

$$60\sqrt{15x} \div 4\sqrt{5x} = 15\sqrt{3}$$

As division is finding a quotient which, multiplied by the divisor, will produce the dividend, the coefficient of the quotient must be a number which, multiplied by 4, will give 60, the coefficient of the dividend, i. e. 15; and the radical part of the quotient must be a quantity which, multiplied by $\sqrt{5x}$, will give $\sqrt{15x}$, i. e. $\sqrt{3}$; the quotient required, therefore, is $15\sqrt{3}$.

2. Divide $6\sqrt{4y}$ by $2\sqrt[3]{2y}$.

OPERATION.

$$6\sqrt{4y} \div 2\sqrt[3]{2y} = 6\sqrt[6]{64y^3} \div 2\sqrt[6]{4y^2} = 3\sqrt[6]{16y}$$

We reduce the radical parts to equivalent radicals having a common index (Art. 155), and then divide as in the preceding example.

3. Divide $\sqrt{a}$ by $\sqrt[3]{a}$.

OPERATION.

$$a^{\frac{1}{2}} \div a^{\frac{1}{3}} = \sqrt[6]{a^3 \div a^2} = \sqrt[6]{a}, \text{ or } a^{\frac{1}{6}}$$

Dividing as in the preceding examples, we have $\sqrt[6]{a}$, or $a^{\frac{1}{6}}$. But $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$; i. e. the index of the quotient is the index of the dividend minus the index of the divisor.

From these examples we deduce the following

RULE.

I. *Reduce the radical parts, if necessary, to equivalent radicals having a common index, and to the quotient of the radical parts placed under the common radical sign prefix the quotient of their coefficients.*

II. *Roots of the same quantity are divided by subtracting the fractional index of the divisor from that of the dividend.*

4. Divide $16\sqrt{ax}$ by $8\sqrt{a^2x}$. Ans. $2\sqrt{a^{-1}}$.

5. Divide $4\sqrt{a^2 - b^2}$ by $2\sqrt{a - b}$.
Ans. $2\sqrt{a + b}$.

6. Divide $6\sqrt{27}$ by $3\sqrt{3}$. Ans. 6.

7. Divide $\sqrt[n]{x}$ by $\sqrt[m]{x}$. Ans. $\sqrt[mn]{x^{m-n}}$.

8. Divide $\sqrt[m]{a}$ by $\sqrt[m]{b}$. Ans. $\sqrt[m]{\frac{a}{b}}$.

9. Divide 3 by $\sqrt{3}$. Ans. $\sqrt{3}$.

10. Divide x by $\sqrt[3]{x}$. Ans. $\sqrt[3]{x^2}$.

11. Divide $4a^2\sqrt{x}$ by $2a^{-1}\sqrt{y}$. Ans. $2a^3\sqrt{\frac{x}{y}}$.

12. Divide $\sqrt{5}$ by $\sqrt[3]{5}$.

13. Divide $\sqrt[3]{7}$ by $\sqrt[4]{7}$.

14. Divide $\sqrt[3]{a}$ by $\sqrt[5]{a}$.

15. Divide $\frac{3}{5}\sqrt{\frac{x}{y}}$ by $\frac{2}{3}\sqrt[3]{\frac{y}{x}}$.

CASE VIII.

161. To involve radicals.

1. Find the cube of $3\sqrt{x}$.

OPERATION.

$$(3\sqrt{x})^3 = 3\sqrt{x} \times 3\sqrt{x} \times 3\sqrt{x}$$
$$= 27\sqrt{x^3} = 27\,x\sqrt{x}$$

In accordance with the definition of involution, we take the quantity three times as a factor. By Art. 159 the product is $27\sqrt{x^3}$.

2. Find the square of $2\sqrt[3]{a}$.

OPERATION.

$$(2\sqrt[3]{a})^2 = (2a^{\frac{1}{3}})^2 = 4a^{\frac{2}{3}}$$

In this case we have used the fractional exponent, and found the square of the given quantity by multiplying its exponent by the index of the required power, according to Art. 126. Hence,

RULE.

I. *Involve the radical as if it were rational, and placing it under its proper radical sign, prefix the required power of its coefficient.*

II. *A radical can be involved by multiplying its fractional exponent by the index of the required power.*

NOTE.—Dividing the index of the root is the same as multiplying the fractional exponent. Thus the square of $\sqrt[6]{a}$ is $\sqrt[3]{a}$. For $(a^{\frac{1}{6}})^2 = a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

3. Find the cube of $3x\sqrt{a}$.

Ans. $27ax^3\sqrt{a}$, or $27a^{\frac{3}{2}}x$.

4. Find the square of $4a^{\frac{1}{3}}$. Ans. $16a^{\frac{2}{3}}$, or $16\sqrt[3]{a^2}$.

5. Find the fourth power of $3\sqrt{x}$. Ans. $81x^2$.

6. Find the nth power of $a\sqrt[4]{x}$. Ans. $a^n\sqrt[4]{x^n}$.

7. Find the fourth power of $5\sqrt{\frac{1}{5}}$. Ans. 25.

8. Find the cube of $3\sqrt{7}$. Ans. $189\sqrt{7}$.

9. Find the fourth power of $\dfrac{a^{\frac{1}{2}}b^2}{x^{-2}y^{\frac{1}{4}}}$.

10. Find the cube of $2\sqrt{4x}$.

CASE IX.

162. To evolve radicals.

1. Find the cube root of $8a^3\sqrt[5]{a^3x^3}$.

OPERATION.

$$\sqrt[3]{8a^3\sqrt[5]{a^3x^3}} = 2a\sqrt[5]{ax}$$

As the root of the product is equal to the product of the roots (Art. 143, Note 2), we prefix to the cube root of the radical part the cube root of the rational part. The cube root of the radical part must be a quantity which, taken three times as a factor, will produce $\sqrt[5]{a^3x^3}$; i. e. $\sqrt[5]{ax}$.

2. Find the fourth root of $\sqrt[3]{x}$.

OPERATION.

$$\sqrt[4]{\sqrt[3]{x}} = \left(x^{\frac{1}{3}}\right)^{\frac{1}{4}} = x^{\frac{1}{12}}, \text{ or } \sqrt[12]{x}$$

In this case we have used the fractional exponent, and found the fourth root by dividing the exponent of the given quantity by the index of the required root, according to Art. 143. Hence,

RULE.

I. *Evolve the radical as if it were rational, and, placing it under its proper radical sign, prefix the required root of its coefficient.*

II. *A radical can be evolved by dividing its fractional exponent by the index of the required root.*

NOTE. — Multiplying the index of the root is the same as dividing the fractional exponent. Thus, the square root of $\sqrt[3]{a}$ is $\sqrt[6]{a}$. For $(a^{\frac{1}{3}})^{\frac{1}{2}} = a^{\frac{1}{6}}$, or $\sqrt[6]{a}$.

3. Find the square root of $5a\sqrt[3]{4x}$.

$$(5a\sqrt[3]{4x})^{\frac{1}{2}} = (\sqrt[3]{500a^3x})^{\frac{1}{2}} = \sqrt[6]{500a^3x}, \text{ Ans.}$$

4. Find the cube root of $x^{-1}\sqrt[3]{a^2b}$. Ans. $\sqrt[9]{\frac{a^2b}{x^3}}$.

5. Find the fifth root of $x^2\sqrt{x}$. Ans. $\sqrt{x}$.

6. Find the fourth root of $\frac{1}{4}\sqrt{\frac{1}{2}}$. Ans. $\sqrt[8]{\frac{1}{32}}$.

7. Find the cube root of $7\sqrt{3}$. Ans. $\sqrt[6]{147}$.

8. Find the square root of $12\sqrt{5}$.

POLYNOMIALS HAVING RADICAL TERMS.

163. It appears from the principles already established, that the laws which apply to calculations with quantities which have exponents, apply equally well whether the exponents are positive or negative, integral or fractional. The following examples, therefore, can be done by rules already given.

1. Add $4a - 3\sqrt{y}$ and $3a + 2\sqrt{y}$. Ans. $7a - \sqrt{y}$.

2. Add $3x + \sqrt[3]{135}$ and $7x - \sqrt[3]{1080}$.
Ans. $10x - 3\sqrt[3]{5}$.

3. Add $2\sqrt{28} - \sqrt{27}$ and $2\sqrt{63} + \sqrt{48}$.

4. Subtract $15x - \sqrt{50a}$ from $13x - \sqrt{8a}$.
Ans. $3\sqrt{2a} - 2x$.

5. Subtract $\sqrt{ax^2} - \sqrt{4b}$ from $\sqrt{ax} - \sqrt{16b}$.
Ans. $\sqrt{ax} - x\sqrt{a} - 2\sqrt{b}$.

6. Subtract $\sqrt[3]{32} - \sqrt{242}$ from $-3\sqrt[3]{4} - 7\sqrt{3}$.

7. Multiply $\sqrt{a} - \sqrt{b}$ by $\sqrt{a} - \sqrt{x}$.

OPERATION.

$$\begin{array}{l}
\sqrt{a} - \sqrt{b} \\
\underline{\sqrt{a} - \sqrt{x}\qquad\qquad\qquad} \\
\quad a - \sqrt{ab} - \sqrt{ax} + \sqrt{bx}
\end{array}$$

8. Multiply $xy + \sqrt{ab}$ by $4 - \sqrt{ab}$.
Ans. $4xy + (4 - xy)\sqrt{ab} - ab$.

9. Multiply $7 + \sqrt{10}$ by $6 - \sqrt{10}$.
Ans. $32 - \sqrt{10}$.

10. Multiply $\sqrt{a} + \sqrt{b}$ by $\sqrt{a} - \sqrt{b}$. Ans. $a - b$.

11. Multiply $\sqrt{5} - 4\sqrt[3]{3}$ by $\sqrt{45} + \sqrt[3]{9}$.

12. Multiply $\frac{1}{4}\sqrt{\frac{1}{2}} + 7\sqrt{3}$ by $\frac{1}{4}\sqrt{\frac{1}{2}} - 7\sqrt{3}$.

13. Divide $\sqrt{ax} + \sqrt{ay} + x + \sqrt{xy}$ by $\sqrt{a} + \sqrt{x}$.

OPERATION.

$$\begin{array}{l}
\sqrt{a} + \sqrt{x})\ \sqrt{ax} + \sqrt{ay} + x + \sqrt{xy}\ (\sqrt{x} + \sqrt{y} \\
\qquad\qquad\ \underline{\sqrt{ax} \qquad\qquad + x\qquad\qquad} \\
\qquad\qquad\qquad\qquad \sqrt{ay} \qquad + \sqrt{xy} \\
\qquad\qquad\qquad\qquad \underline{\sqrt{ay} \qquad + \sqrt{xy}}
\end{array}$$

14. Divide $\sqrt{ac} - \sqrt{ad} - \sqrt{bc} + \sqrt{bd}$ by $\sqrt{c} - \sqrt{d}$.
Ans. $\sqrt{a} - \sqrt{b}$.

15. Divide $a^{\frac{1}{2}} x + b^{\frac{1}{2}} x + a^{\frac{1}{2}} y^{\frac{1}{2}} + b^{\frac{1}{2}} y^{\frac{1}{2}}$ by $x + y^{\frac{1}{2}}$.

16. Divide $x - y$ by $\sqrt{x} - \sqrt{y}$. Ans. $\sqrt{x} + \sqrt{y}$.

17. Divide $4xy + 4\sqrt{ab} - xy\sqrt{ab} - ab$ by $4 - \sqrt{ab}$.

18. Expand $(\sqrt{x} + \sqrt{y})^2$. Ans. $x + 2\sqrt{xy} + y$.

19. Expand $(a^{\frac{1}{2}} - b^{-\frac{1}{2}})^2$. Ans. $a - 2\sqrt{\frac{a}{b}} + \frac{1}{b}$.

20. Expand $(\sqrt{a} - \sqrt{b})^4$.
Ans. $a^2 - 4a^{\frac{3}{2}} b^{\frac{1}{2}} + 6ab - 4a^{\frac{1}{2}} b^{\frac{3}{2}} + b^2$.

21. Expand $(4 - \sqrt{3})^3$. Ans. $100 - 51\sqrt{3}$.

22. Expand $(a^{-\frac{1}{2}} - x^{-1})^3$.
Ans. $a^{-\frac{3}{2}} - 3a^{-1}x^{-1} + 3a^{-\frac{1}{2}}x^{-2} - x^{-3}$.

23. Expand $\left(\frac{1}{2} - \sqrt{\frac{1}{a}}\right)^4$.
Ans. $\frac{1}{16} - \frac{1}{2\sqrt{a}} + \frac{3}{2a} - \frac{2}{a\sqrt{a}} + \frac{1}{a^2}$.

24. Expand $\left(\sqrt{\frac{x}{2}} - \sqrt{\frac{y}{3}}\right)^4$.
Ans. $\frac{x^2}{4} - \frac{2x^{\frac{3}{2}}y^{\frac{1}{2}}}{\sqrt{6}} + xy - \frac{4x^{\frac{1}{2}}y^{\frac{3}{2}}}{3\sqrt{6}} + \frac{y^2}{9}$.

25. Find the square root of $a - 2a^{\frac{1}{2}} b^{\frac{2}{3}} + b^{\frac{4}{3}}$.
Ans. $\sqrt{a} - \sqrt[3]{b^2}$.

26. Find the cube root of $x^3 - 3x^2 y^{\frac{1}{3}} + 3xy^{\frac{2}{3}} - y$.
Ans. $x - y^{\frac{1}{3}}$.

27. Find the fourth root of $16a - 32a^{\frac{3}{4}}y^{\frac{2}{3}} + 24a^{\frac{1}{2}}y^{\frac{4}{3}} - 8a^{\frac{1}{4}}y^2 + y^{\frac{8}{3}}$.
Ans. $2a^{\frac{1}{4}} - y^{\frac{2}{3}}$.

SECTION XVIII.

PURE EQUATIONS

WHICH REQUIRE IN THEIR REDUCTION EITHER INVOLUTION OR EVOLUTION.

164. A PURE EQUATION is one that contains but *one* power of the unknown quantity; as,

$$\sqrt{x} + a c = b, \; 4 x^2 + 3 = 7, \text{ or } 14 x^n = a b.$$

165. A PURE QUADRATIC EQUATION is one that contains *only* the second power of the unknown quantity; as,

$$6 x^2 - 14 a = 51 b, \; a y^2 = 13 c d, \text{ or } a c z^2 = 14.$$

166. RADICAL EQUATIONS, i. e. equations containing the unknown quantity under the radical sign, require Involution in their reduction.

167. To reduce radical equations.

1. Reduce $\sqrt{x} - 3 = 8$.

OPERATION.

$$\sqrt{x} - 3 = 8$$

Transposing, $\sqrt{x} = 11$

Squaring, $x = 121$

2. Reduce $\sqrt[3]{x - 4} + 7 = 10$.

OPERATION.

$$\sqrt[3]{x - 4} + 7 = 10$$

Transposing, $\sqrt[3]{x - 4} = 3$

Cubing, $x - 4 = 27$

Transposing, $x = 31$

3. Reduce $\dfrac{\sqrt{d^2+\sqrt{x}}}{\sqrt{a}} = \sqrt{a}$.

OPERATION.

$$\frac{\sqrt{d^2+\sqrt{x}}}{\sqrt{a}} = \sqrt{a}$$

Clearing of fractions, $\sqrt{d^2+\sqrt{x}} = a$

Squaring, $d^2 + \sqrt{x} = a^2$

Transposing, $\sqrt{x} = a^2 - d^2$

Squaring, $x = (a^2 - d^2)^2$

Hence, to reduce radical equations, we deduce from these examples the following general

RULE.

Transpose the terms so that a radical part shall stand by itself; then involve each member of the equation to a power of the same name as the root; if the unknown quantity is still under the radical sign, transpose and involve as before; finally reduce as usual.

4. Reduce $4 + \frac{3}{4} + 3\sqrt{x} = \frac{67}{4}$. Ans. $x = 16$.

5. Reduce $\frac{1}{4}\sqrt{\frac{8}{x}} = \frac{1}{2}$. Ans. $x = 2$.

6. Reduce $(\sqrt{x} + 4)^{\frac{1}{4}} = 2$. Ans. $x = 144$.

7. Reduce $\sqrt{11 + x} = \sqrt{x} + 1$. Ans. $x = 25$.

8. Reduce $\sqrt{x - 7} = \sqrt{x + 18} - \sqrt{5}$.
Ans. $x = 27$.

9. Reduce $\dfrac{\sqrt{x}}{x - cx} = \dfrac{x}{\sqrt{x}}$. Ans. $x = \dfrac{1}{1 - c}$.

10. Reduce $\dfrac{\sqrt{x} - 2}{\sqrt{x} + 10} = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 23}$. Ans. $x = 9$.

170. Equations containing radical quantities may require in their reduction both Involution and Evolution; and in this case the rule in Art. 167, as well as that in Art. 169, must be applied. Which rule is first to be applied depends upon whether the expression containing the unknown quantity is evolved or involved.

1. Reduce $17 - \sqrt{x^3 - 2} = 12$.

OPERATION.

	$17 - \sqrt{x^3 - 2} = 12$
Transposing, &c.,	$\sqrt{x^3 - 2} = 5$
Squaring,	$x^3 - 2 = 25$
Transposing and uniting,	$x^3 = 27$
Extracting the cube root,	$x = 3$

2. Reduce $(\sqrt{x^3 - 4} + 3)^3 = 125$. Ans. $x = 2$.

3. Reduce $\sqrt{\dfrac{10x^2 - 2}{2x}} = \sqrt{x}$. Ans. $x = \pm \frac{1}{2}$.

4. Reduce $\sqrt{x + a} = \dfrac{a + b}{\sqrt{x - a}}$.

Ans. $x = \pm \sqrt{2a^2 + 2ab + b^2}$.

5. Reduce $\frac{49}{2} - \sqrt[3]{3\left(x^4 + \frac{11}{81}\right)} = \frac{47}{2}$.

6. Reduce $\sqrt[4]{2x^4 + 8x^3 + 24x^2 + 32x} = x + 2$.

7. Reduce $\sqrt[3]{9(x^4 + 19) + 100} - \frac{7}{2} = \frac{13}{2}$.

171. Equations which contain two or more unknown quantities may require for their reduction involution, or evolution, or both. In these equations the elimination is effected by the same principles as in simple equations. (Arts. 112–114.)

1. Given $\left\{\begin{array}{l} \frac{3x^2}{5} - \frac{y}{4} = 14 \\ 2x^2 + y = 54 \end{array}\right\}$, to find x and y.

OPERATION.

$$\frac{3x^2}{5} - \frac{y}{4} = 14 \quad (1) \qquad 2x^2 + y = 54 \quad (2)$$

$$\frac{12x^2}{5} - y = 56 \quad (3)$$

$$\frac{22x^2}{5} = 110 \quad (4)$$

$$x^2 = 25 \quad (5)$$

$$x = \pm 5 \quad (6)$$

$$15 - \frac{y}{4} = 14 \quad (7)$$

$$y = 4 \quad (8)$$

Subtracting four times (1) from (2), we obtain (4), which reduced gives (6), or $x = \pm 5$; substituting this value of x in (1), we obtain (7), which reduced gives (8), or $y = 4$.

Find the value of the unknown quantities in the following equations:—

2. Given $\left\{\begin{array}{l} \frac{x+y}{4} = x - y \\ xy = 15 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \pm 5. \\ y = \pm 3. \end{array}\right.$

3. Given $\left\{\begin{array}{l} 3x - 4y = 2y \\ x^2 + 6y^2 = 90 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \pm 6. \\ y = \pm 3. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 4x^3z = 20 \\ 2xz = 10 \\ 3yz = 45 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \pm 1. \\ y = \pm 3. \\ z = \pm 5. \end{array}\right.$

5. Given $\left\{\begin{array}{l} \sqrt[3]{x} - \sqrt[3]{y} = 1 \\ \sqrt[3]{x} + \sqrt[3]{y} = 5 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 27. \\ y = 8. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x^4 + y^4 = 97 \\ x - y = y - 2x \end{array}\right\}$.

7. Given $\left\{\begin{array}{l} \sqrt{\frac{x-y}{y}} = \sqrt{\frac{3}{5}} \\ x^2 - 2y^2 = 14 \end{array}\right\}$.

PROBLEMS

PRODUCING PURE EQUATIONS ABOVE THE FIRST DEGREE.

172. Though the numerical negative values obtained in solving the following Problems satisfy the equations formed in accordance with the given conditions, they are practically inadmissible, and are therefore not given in the answers.

1. A gentleman being asked how many dollars he had in his purse, replied, "If you add 21 to the number and subtract 4 from the square root of the sum, the remainder will be 6." How many had he?

SOLUTION.

Let $x =$ number of dollars.

Then, $\sqrt{x + 21} - 4 = 6$

Transposing, $\sqrt{x + 21} = 10$

Squaring, $x + 21 = 100$

Transposing, $x = 79$, number of dollars.

2. Divide 20 into two parts whose cubes shall be in the proportion of 27 to 8. Ans. 12 and 8.

3. What two numbers are those whose sum is to the less as $8 : 3$, and the sum of whose squares is 136?
Ans. 10 and 6.

4. What number is that whose half multiplied by its third gives 54?

5. What number is that whose fourth and seventh multiplied together gives $46\frac{2}{7}$? Ans. 36.

6. There is a rectangular field containing 4 acres whose length is to its breadth as $8 : 5$. What is its length and breadth?

7. There are two numbers whose sum is 17, and the less divided by the greater is to the greater divided by the less as 64 : 81. What are the numbers?

Ans. 8 and 9.

8. The sum of the squares of two numbers is 65, and the difference of their squares 33. What are the numbers?

9. The sum of the squares of two quantities is a, and the difference of their squares b. What are the quantities?

Ans. $\pm\sqrt{\frac{1}{2}(a+b)}$ and $\pm\sqrt{\frac{1}{2}(a-b)}$.

10. A gentleman sold two fields which together contained 240 acres. For each he received as many dollars an acre as there were acres in the field, and what he received for the larger was to what he received for the smaller as 49 : 25. What are the contents of each?

Ans. Larger, 140; smaller, 100 acres.

11. What are the two quantities whose product is a and quotient b?

Ans. $\pm\sqrt{ab}$ and $\pm\sqrt{\frac{a}{b}}$.

12. What two numbers are as $m : n$, the sum of whose squares is a?

Ans. $\pm\frac{m\sqrt{a}}{\sqrt{(m^2+n^2)}}$ and $\pm\frac{n\sqrt{a}}{\sqrt{(m^2+n^2)}}$.

13. What two numbers are as $m : n$, the difference of whose squares is a?

Ans. $\pm\frac{m\sqrt{a}}{\sqrt{(m^2-n^2)}}$ and $\pm\frac{n\sqrt{a}}{\sqrt{(m^2-n^2)}}$.

14. Several gentlemen made an excursion, each taking \$484. Each had as many servants as there were gentlemen, and the number of dollars which each had was four times the number of all the servants. How many gentlemen were there?

Ans. 11.

15. Find three numbers such that the product of the first and second is 12; of the second and third, 20; and the sum of the squares of the first and third, 34.

SECTION XIX.

AFFECTED QUADRATIC EQUATIONS.

173. AN AFFECTED QUADRATIC EQUATION is one that contains both the first and second powers of the unknown quantity; as,

$$3x^2 - 4x = 16; \text{ or } ax - bx^2 = c.$$

174. Every affected quadratic equation can be reduced to the form

$$x^2 + bx = c,$$

in which b and c represent any quantities whatever, positive or negative, integral or fractional.

For all the terms containing x^2 can be collected into one term whose coefficient we will represent by a; all the terms containing x can be collected into one term whose coefficient we will represent by d; and all the other terms can be united, whose aggregate we will represent by e. Therefore every affected quadratic equation can be reduced to the form

$$ax^2 + dx = e \qquad (1)$$

Dividing (1) by a, $$x^2 + \frac{d}{a}x = \frac{e}{a} \qquad (2)$$

Letting $\frac{d}{a} = b$, and $\frac{e}{a} = c$, we have $$x^2 + bx = c \qquad (3)$$

175. The first member of the equation $x^2 + bx = c$ cannot be a perfect square. (Art. 145, Note 2.) But we know that the square of a binomial is *the square of the first term plus or minus twice the product of the two terms plus the square of the last term*; and if we can find the third term which will make $x^2 + bx$ a perfect

square of a binomial, we can then reduce the equation $x^2 + bx = c$.

Since bx has in it as a factor the square root of x^2, x^2 can be the first term of the square of a binomial, and bx the second term of the same square; and since the second term of the square is twice the product of the two terms of the binomial, the last term of the binomial must be the quotient arising from dividing the second term of the square by twice the square root of the first term of the square of the binomial; i. e. the last term of the binomial is $\frac{bx}{2x} = \frac{b}{2}$; and therefore the third term of the square must be $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$. Adding $\frac{b^2}{4}$ to each member, we have (2), an equation whose first member is a perfect square. Extracting the square root of each member of (2), and transposing, we obtain (4), or $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c}$, which is a general expression for the value of x in any equation in the form of $x^2 + bx = c$.

OPERATION.

$$x^2 + bx = c \qquad (1)$$

$$x^2 + bx + \frac{b^2}{4} = \frac{b^2}{4} + c \qquad (2)$$

$$x + \frac{b}{2} = \pm \sqrt{\frac{b^2}{4} + c} \qquad (3)$$

$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c} \qquad (4)$$

Hence, as every affected quadratic equation can be reduced to the form $x^2 + bx = c$, in which b and c represent any quantities whatever, positive or negative, integral or fractional, every affected quadratic equation can be reduced by the following

RULE.

Reduce the equation to the form $x^2 + bx = c$, and add to each member the square of half the coefficient of x.

Extract the square root of each member, and then reduce as in simple equations.

1. Reduce $7x^2 - 28x + 14 = 238$.

OPERATION.

$$7x^2 - 28x + 14 = 238$$

Transposing, $7x^2 - 28x = 224$

Dividing by 7, $x^2 - 4x = 32$

Completing the square, $x^2 - 4x + 4 = 36$

Evolving, $x - 2 = \pm 6$

Transposing, $x = 2 \pm 6 = 8$, or -4

NOTE. — Since in reducing the general equation $x^2 + bx = c$ we find $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c}$, every affected quadratic equation must have two roots; one obtained by considering the expression $\sqrt{\frac{b^2}{4} + c}$ positive, the other by considering this expression negative. Whenever $\sqrt{\frac{b^2}{4} + c} = 0$ these two roots will be equal.

2. Reduce $\frac{x^2}{5} - \frac{x}{10} + \frac{13}{20} = \frac{x^2}{2} + \frac{x}{4}$.

OPERATION.

$$\frac{x^2}{5} - \frac{x}{10} + \frac{13}{20} = \frac{x^2}{2} + \frac{x}{4}$$

Clearing of fractions, $4x^2 - 2x + 13 = 10x^2 + 5x$

Transposing, $-6x^2 - 7x = -13$

Dividing by -6, $x^2 + \frac{7x}{6} = \frac{13}{6}$

Completing the square, $x^2 + (\ ) + \frac{49}{144} = \frac{49}{144} + \frac{13}{6} = \frac{361}{144}$

Evolving, $x + \frac{7}{12} = \pm \frac{19}{12}$

Transposing, $x = -\frac{7}{12} \pm \frac{19}{12} = 1$, or $-2\frac{1}{6}$

NOTE.—In completing the square, as the second term disappears when the root is extracted, we have written ( ) in place of it.

3. Reduce $3x^2 - 25 + 6x = 80$.
Ans. $x = 5$, or -7.

4. Reduce $x - \frac{24 - x}{x} = 3$. Ans. $x = 6$, or -4.

5. Reduce $2x + \frac{x+1}{x-1} = 7$. Ans. $x = 2$.

NOTE.—In this example *both* roots are 2.

6. Reduce $7x - \frac{x^2 + 4}{x - 4} = 5x - 1$.
Ans. $x = 8$, or -1.

7. Reduce $17 - \frac{14 - x}{7} = \frac{13 - x}{8 - x} + 10$.
Ans. $x = 7$, or -27.

8. Reduce $\frac{x}{x+5} + \frac{1}{3} = \frac{x}{10}$. Ans. $x = 10$, or $-1\frac{2}{3}$.

9. Reduce $\frac{12}{x} + \frac{80 - 4x^2}{x^2} = 4$.

10. Reduce $\frac{16}{x} - \frac{100 - 9x}{4x^2} = 3$.

176. Whenever an equation has been reduced to the form $x^2 + bx = c$, its roots can be written at once; for this equation reduced (Art. 175) gives $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c}$. Hence,

The roots of an equation reduced to the form $x^2 + bx = c$ are equal to one half the coefficient of x with the opposite sign, plus or minus the square root of the sum of the square of one half this coefficient and the second member of the equation.

In accordance with this, find the roots of x in the following equations:—

1. Reduce $x^2 + 8x = 65$.

$$x = -4 \pm \sqrt{16 + 65} = 5, \text{ or } -13, \textit{ Ans.}$$

2. Reduce $x^2 - 10x = -24$.

$$x = 5 \pm \sqrt{25 - 24} = 6, \text{ or } 4, \textit{ Ans.}$$

3. Reduce $x^2 - 6x = -5$. Ans. $x = 5$, or 1.

4. Reduce $x^2 + 7x = 170$.

$$x = -\frac{7}{2} \pm \sqrt{\frac{49}{4} + 170} = 10, \text{ or } -17, \textit{ Ans.}$$

5. Reduce $x^2 + \frac{1}{2}x = \frac{1}{2}$.

$$x = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}} = \frac{1}{2}, \text{ or } -1, \textit{ Ans.}$$

6. Reduce $x^2 + \frac{1}{8}x = \frac{9}{8}$. Ans. $x = 1$, or $-1\frac{1}{8}$.

7. Reduce $x^2 - \frac{4}{5}x = -\frac{3}{25}$. Ans. $x = \frac{3}{5}$, or $\frac{1}{5}$.

8. Reduce $\frac{x^2}{7} = \frac{x}{4} + 5\frac{1}{4}$. Ans. $x = 7$, or $-5\frac{1}{4}$.

SECOND METHOD OF COMPLETING THE SQUARE.

177. The method already given for completing the square can be used in all cases; but it often leads to inconvenient fractions. The more difficult fractions are introduced by dividing the equation by the coefficient of x^2, to reduce it to the form $x^2 + bx = c$. To present a method of completing the square without introducing these fractions, we will reduce equation (1) in Art. 174.

1. Reduce $a x^2 + d x = e$.

OPERATION.

$$a x^2 + d x = e \qquad (1)$$

$$a^2 x^2 + a d x = a e \qquad (2)$$

$$a^2 x^2 + a d x + \frac{d^2}{4} = \frac{d^2}{4} + a e \qquad (3)$$

$$a x + \frac{d}{2} = \pm \sqrt{\frac{d^2}{4} + a e} \qquad (4)$$

$$x = \frac{1}{a}\left(-\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + a e}\right) \qquad (5)$$

Multiplying (1) by a, the coefficient of x^2, we obtain (2), in which the first term must be a perfect square. Since $a d x$, the second term, has in it as a factor the square root of $a^2 x^2$, $a^2 x^2$ can be the first term of the square of a binomial, and $a d x$ the second term; and since the second term of the square is twice the product of the two terms of the binomial, the last term of the binomial must be the second term of the square divided by twice the square root of the first term of the square of the binomial, or $\frac{a d x}{2 a x} = \frac{d}{2}$; and therefore the term required to complete the square is $\frac{d^2}{4}$, which is the square of one half of the coefficient of x in (1). Adding $\frac{d^2}{4}$ to both members of (2), we obtain (3), whose first member is the square of a binomial. Extracting the square root of (3) and reducing, we obtain (5), or $x = \frac{1}{a}\left(-\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + a e}\right)$.

Hence, to reduce an affected quadratic equation, we have this second

RULE.

Reduce the equation to the form $a x^2 + d x = e$; then multiply the equation by the coefficient of x^2, and add to each member the square of half the coefficient of x.

Extract the square root of each member, and then reduce as in simple equations.

NOTE 1.—This method does not introduce fractions into the equation when the numerical part of the coefficient of x is even. When the coefficient of x^2 is unity, this method becomes the same as the first method.

NOTE 2.—If the coefficient of x^2 is already a perfect square the square can be completed without multiplying the equation, by *adding to both members the square of the quotient arising from dividing the second term by twice the square root of the first.* This method also becomes the same as the first method when the coefficient of x^2 is unity.

NOTE 3.—As an even root of a negative quantity is impossible or imaginary, the sign of the first term, if it is not positive, must be made so by changing the signs of all the terms of the equation.

2. Reduce $3x^2 + 8x = 28$.

OPERATION.

$$3x^2 + 8x = 28$$

Completing the square, $9x^2 + (\ ) + 16 = 16 + 84 = 100$

Extracting square root, $3x + 4 = \pm 10$

Whence, $3x = -4 \pm 10 = 6, \text{ or } -14$

And $x = 2, \text{ or } -4\tfrac{2}{3}$

3. Reduce $25x^2 - 10x = 195$.

OPERATION.

$$25x^2 - 10x = 195$$

Completing sq. by Note 2, $25x^2 - (\ ) + 1 = 1 + 195 = 196$

Extracting square root, $5x - 1 = \pm 14$

Whence, $5x = 1 \pm 14 = 15, \text{ or } -13$

And $x = 3, \text{ or } -2\tfrac{3}{5}$

4. Reduce $5x^2 - 20x = -15$. Ans. $x = 3$, or 1.

5. Reduce $7x^2 - 8x = 12\tfrac{4}{7}$. Ans. $x = \tfrac{4}{7} \pm \tfrac{2}{7}\sqrt{26}$.

6. Reduce $7x^2 - 4ax = \dfrac{5a^2}{7}$. Ans. $x = \dfrac{5a}{7}$, or $-\dfrac{a}{7}$.

7. Reduce $\dfrac{4x}{14 - x} - \dfrac{x - 1}{3} = x - 3$. Ans. $x = 10$, or $3\tfrac{1}{2}$.

8. Reduce $x^2 + \frac{17x}{4} = -\frac{17x}{4} - 4$.

9. Reduce $\frac{x^2 - 4}{7} - \frac{x - 3}{6} = \frac{3x - 7}{2}$.

THIRD METHOD OF COMPLETING THE SQUARE.

178. The method of the preceding Article introduces fractions whenever the numerical coefficient of x is not even. To present a method of completing the square without introducing any fraction, we will again reduce equation (1) in Art. 174.

1. Reduce $a x^2 + d x = e$.

OPERATION.

$$a x^2 + d x = e \qquad (1)$$

$$x^2 + \frac{d}{a} x = \frac{e}{a} \qquad (2)$$

$$x^2 + \frac{d}{a} x + \frac{d^2}{4 a^2} = \frac{d^2}{4 a^2} + \frac{e}{a} \qquad (3)$$

$$4 a^2 x^2 + 4 a d x + d^2 = d^2 + 4 a e \qquad (4)$$

$$2 a x + d = \pm \sqrt{d^2 + 4 a e} \qquad (5)$$

$$x = \frac{-d \pm \sqrt{d^2 + 4 a e}}{2 a} \qquad (6)$$

Dividing (1) by a, the coefficient of x^2, we have (2); then completing the square according to the Rule in Art. 175, we have (3); and if we multiply (3) by $4 a^2$, it will give (4), an equation free from fractions (unless a, d, or e in (1) are themselves fractions), and one whose first member is the square of a binomial. To produce this equation directly from (1), we have only to multiply (1) by $4 a$; i. e. by four times the coefficient of x^2, and add to both members d^2; i. e. the square of the coefficient of x. Reducing we have (6), which is a general expression for the value of x in any equation in the form of $a x^2 + d x = e$.

Hence, to reduce an affected quadratic equation, we have this third

RULE.

Reduce the equation to the form $a x^2 + d x = e$; *then multiply the equation by four times the coefficient of* x^2 *and add to each member the square of the coefficient of* x.

Extract the square root of each member, and then reduce as in simple equations.

NOTE.—The third Note under the Rule in Art. 177 is applicable in *all* cases.

2. Reduce $5x^2 - 7x = 24$.

OPERATION.

	$5x^2 - 7x = 24$
Multiplying by 5×4 and adding 7^2 to each member,	$100x^2 - (\ ) + 49 = 49 + 480 = 529$
Extracting the square root,	$10x - 7 = \pm 23$
Transposing,	$10x = 7 \pm 23 = 30$, or -16
Whence,	$x = 3$, or -1.6

NOTE.—The multiplication of the coefficient of x^2 need only be expressed. Its coefficient after evolving is double its original coefficient.

3. Reduce $\dfrac{44x^2 - 15x}{7} = 293$.

OPERATION.

	$\dfrac{44x^2 - 15x}{7} = 293$
Clearing of fractions,	$44x^2 - 15x = 2051$
Completing square,	$176 \times 44x^2 - (\ ) + 225 = 225 + 360976 = 361201$
Evolving,	$88x - 15 = \pm 601$
Transposing,	$88x = 15 \pm 601 = 616$, or -586
Whence,	$x = 7$, or $-\dfrac{293}{44}$

4. Reduce $7x^2 - 15x = -2$. Ans. $x = 2$, or $\frac{1}{7}$.

5. Reduce $\frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3$.

Ans. $x = 1$, or -28.

6. Reduce $\frac{10}{x+2} + \frac{9}{x} = 5$. Ans. $x = 3$, or $-1\frac{1}{5}$.

7. Reduce $\frac{6}{x+1} + \frac{2}{x} = 3$. Ans. $x = 2$, or $-\frac{1}{3}$.

8. Reduce $\frac{4x+4}{x} - \frac{3x-3}{2x-1} = \frac{10x+10}{3x}$.

9. Reduce $\frac{1}{7-2x} + \frac{3}{2x+4} = \frac{13}{10}$.

10. Reduce $\sqrt[3]{x^3 - a^3} = x - b$.

Ans. $x = \frac{b}{2} \pm \sqrt{\frac{4a^3 - b^3}{12b}}$.

11. Reduce $5 - 3x^{-1} = 110x^{-2}$.

NOTE. — Multiply by x^2.

12. Reduce $\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$.

NOTE. — Divide by $\sqrt{x}$.

179. The rules which have been given for the solution of affected quadratic equations apply equally well to any equation containing but two powers of the unknown quantity *whenever the index of one power is exactly twice that of the other*. By the same reasoning as in Art. 174, it can be shown that all such equations can be reduced to the form

$$ax^{2n} + dx^n = e,$$

or

$$x^{2n} + bx^n = c.$$

It will be seen that the first member is composed of two terms so related that they may be the first two terms of a binomial square, and we can supply the third by one of the rules already given for completing the square.

1. Reduce $x^6 - 2x^3 = 48$.

OPERATION.

$$x^6 - 2x^3 = 48 \qquad (1)$$

$$x^6 - 2x^3 + 1 = 1 + 48 = 49 \qquad (2)$$

$$x^3 - 1 = \pm 7 \qquad (3)$$

$$x^3 = 8, \text{ or } -6 \qquad (4)$$

$$x = 2, \text{ or } \sqrt[3]{-6} \qquad (5)$$

Since the square root of x^6 is x^3, it is evident that the second term contains as one of its factors the square root of the first term; i. e. the first member of the equation is composed of two terms so related that they may be the first two terms of the square of a binomial. Completing the square, we have (2); extracting the square root of each member of (2), we obtain (3); transposing we have (4), and extracting the cube root of (4) we have $x = 2$, or $\sqrt[3]{-6}$.

2. Reduce $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 160$.

OPERATION.

$$3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 160 \qquad (1)$$

$$36x^{\frac{3}{2}} - (\;) + 16 = 16 + 1920 = 1936 \qquad (2)$$

$$6x^{\frac{3}{4}} - 4 = \pm 44 \qquad (3)$$

$$6x^{\frac{3}{4}} = 48, \text{ or } -40 \qquad (4)$$

$$x^{\frac{3}{4}} = 8, \text{ or } -\tfrac{20}{3} \qquad (5)$$

$$x^{\frac{1}{4}} = 2, \text{ or } \sqrt[3]{-\tfrac{20}{3}} \qquad (6)$$

$$x = 16, \text{ or } (-\tfrac{20}{3})^{\frac{4}{3}} \qquad (7)$$

In this equation the index of the higher power is exactly twice that of the lower. Completing the square we have (2); extracting the square root of each member of (2), we have (3); transposing, we have (4), which divided by 6 gives (5); extracting the cube root of (5), we have (6), which involved to the fourth power gives (7).

3. Reduce $\frac{x^6}{2} + \frac{x^3}{4} = \frac{3}{32}$. Ans. $x = \sqrt[3]{\frac{1}{4}}$, or $-\sqrt[3]{\frac{3}{4}}$.

4. Reduce $\sqrt[3]{x^2} + \frac{3}{2}\sqrt[3]{x} = 1$. Ans. $x = \frac{1}{8}$, or -8.

5. Reduce $x - \frac{2}{3}\sqrt{x} = 44\frac{1}{3}$. Ans. $x = 49$, or $40\frac{1}{9}$.

6. Reduce $x^5 - x^{\frac{5}{2}} = 0$. Ans. $x = 1$, or 0.

7. Reduce $3x^4 - 2x^2 + 3 = 228$.
Ans. $x = \pm 3$, or $\pm 5\sqrt{-\frac{1}{3}}$.

8. Reduce $3x^{2n} - 2x^n = 8$.
Ans. $x = \sqrt[n]{2}$, or $\sqrt[n]{-\frac{4}{3}}$.

9. Reduce $\frac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \frac{4 + \sqrt{x}}{\sqrt{x}}$. Ans. $x = 64$, or 4.

10. Reduce $x^4 - ax^2 = b$.
Ans. $x = \pm\sqrt{\left(\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b}\right)}$.

180. A polynomial may take the place of the unknown quantity in an affected quadratic equation. In this case the equation can be reduced by considering the polynomial as a single quantity.

1. Reduce $(x - 4)^2 - 2(x - 4) = 8$.

OPERATION.

$$(x - 4)^2 - 2(x - 4) = 8 \qquad (1)$$
$$(x - 4)^2 - (\ ) + 1 = 1 + 8 = 9 \qquad (2)$$
$$x - 4 - 1 = \pm 3 \qquad (3)$$
$$x = 5 \pm 3 = 8, \text{ or } 2 \qquad (4)$$

Considering $x - 4$ as a single term, and completing the square, we have (2); extracting the square root, transposing, &c., we have (4), or $x = 8$, or 2.

NOTE.—We might put $(x - 4) = y$; then $(x - 4)^2 = y^2$, and the equation becomes $y^2 - 2y = 8$. After finding the value of y in this equation, $x - 4$ must be substituted for y.

2. Reduce $\sqrt{5 + x} + \sqrt[4]{5 + x} = 6$.
Ans. $x = 11$, or 76.

3. Reduce $4 + 2x - x^2 + \frac{1}{2}\sqrt{4 + 2x - x^2} = \frac{1}{2}$.
Ans. $x = 1 \pm \frac{1}{2}\sqrt{19}$, or 3, or -1.

4. Reduce $x + 7 - 7\sqrt{x+7} = 8 - 5\sqrt{x+7}$.

Ans. $x = 9$, or -3.

5. Reduce $(x-5)^2 - 3\sqrt{x-5} = \frac{40}{x-5}$.

Ans. $x = 9$, or $5 + \sqrt[3]{25}$.

6. Reduce $x^2 + 3x + \sqrt{x^2 + 3x + 6} = 14$.

NOTE. — Add 6 to both members.

Ans. $x = 2$, or -5, or $-\frac{3}{2} \pm \frac{1}{2}\sqrt{85}$.

7. Reduce $4 + x^2 - 2x - 2\sqrt{6 - 2x + x^2} = 1$.

Ans. $x = 3$, or -1, or $1 \pm 2\sqrt{-1}$.

8. Reduce $\sqrt{x^2 + x + 6} = \frac{60}{\sqrt{x^2 + x + 6}} - 4$.

Ans. $x = 5$, or -6, or $-\frac{1}{2} \pm \frac{1}{2}\sqrt{377}$.

181. Of the methods given for completing the square, the first is the best when the coefficient of the less power of the unknown quantity is even, and the coefficient of the higher power is unity, or when these become so by reduction; the second method is better than the third whenever the coefficient of the less power of the unknown quantity is even. When the equation cannot be reduced by the first method without introducing fractions, if the coefficient of the higher power of the unknown quantity is a perfect square, and the coefficient of the less power is divisible without remainder by twice the square root of the coefficient of the higher power, the method given in Note 2, Art. 177, is the best. Let each of the following equations be reduced by the method best adapted to it.

1. Reduce $4x^2 - 14 = 3x^2 - 12x - 1$.

Ans. $x = 1$, or -13.

2. Reduce $36x^2 + 24x = 1020$.

Ans. $x = 5$, or $-5\frac{2}{3}$.

3. Reduce $x - \frac{1}{7} + \frac{3}{x} = 21.$ Ans. $x = 21$, or $\frac{1}{7}$.

4. Reduce $\frac{8 - x}{2} - \frac{x - 2}{6} = \frac{2x - 11}{x - 3}.$

5. Reduce $\frac{x^2}{2} - \frac{x}{5} - \frac{1}{10} = 23.$

6. Reduce $\frac{x - 3c}{d} = \frac{9(d - c)}{x}.$

Ans. $x = 3(c - d)$, or $3d$.

7. Reduce $\frac{x + 7}{x - 7} = \frac{x - 7}{x + 7} + 2\frac{2}{3}.$

8. Reduce $\frac{3x - 4}{x - 4} = 9 - \frac{x - 2}{2}.$

9. Reduce $\frac{10}{x} - \frac{22}{9} = \frac{14 - 2x}{x^2}.$

10. Reduce $\frac{x}{a} + \frac{a}{x} = \frac{2}{a}.$ Ans. $x = 1 \pm \sqrt{1 - a^2}.$

11. Reduce $3x + 3 = 13 + \frac{x^2}{4}.$

12. Reduce $5x - \frac{3x - 3}{x - 3} = 2x + \frac{3x - 6}{2}.$

13. Reduce $\frac{x + 4}{3} - \frac{4x + 7}{9} + 1 = \frac{7 - x}{x - 3}.$

14. Reduce $2\sqrt{x} - \sqrt{x - 7} = 5.$

Ans. $x = 16$, or $7\frac{1}{9}$.

15. Reduce $2\sqrt{x - a} + 3\sqrt{2x} = \frac{7a + 5x}{\sqrt{x - a}}.$

Ans. $x = 9a$, or $-a$.

16. Reduce $4\sqrt{x} - \sqrt{2x + 1} = \frac{15}{\sqrt{2x + 1}}.$

Ans. $x = 4$, or $-2\frac{2}{7}$.

17. Reduce $5\sqrt{25 - x} = 6\sqrt{25 - x} + x - 13.$

Ans. $x = 16$, or 9.

18. Reduce $\frac{x+4}{2} = \sqrt{4 + \sqrt{2x^3 + x^2}}$.
Ans. $x = 12$, or 4.

19. Reduce $6 + 4x^{-1} - 12x^{-2} = 100x^{-2}$.

20. Reduce $\frac{1}{2}\sqrt[5]{x^2} + 6\sqrt[5]{x} - \frac{13}{7} = \frac{85}{7}$.

21. Reduce $3x^4 - 24x^2 - 80 = 304$.
Ans. $x = \pm 4$, or $\pm 2\sqrt{-2}$.

22. Reduce $\frac{x^6}{3} + 10 = 1 + 4x^3$.
Ans. $x = \sqrt[3]{9}$, or $\sqrt[3]{3}$.

23. Reduce $5x^4 - 3x^2 + \frac{81}{4} = 27$.
Ans. $x = \pm \frac{1}{2}\sqrt{6}$, or $\pm 3\sqrt{-\frac{1}{10}}$.

24. Reduce $2x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 4 = 2$.

25. Reduce $6x^{\frac{4}{3}} + 1184 = 5x^{\frac{8}{3}}$.

26. Reduce $\sqrt{x+3} - \sqrt[4]{x+3} = 2$.
Ans. $x = 13$, or -2.

27. Reduce $x^2 - \sqrt{x^2 + x - 5} = 25 - x$.

NOTE. — By transposing $-x$ and subtracting 5 from each member, make the expression without the radical in the first member like that under the radical; then complete the square, &c.

28. Reduce $x^2 - 2x + 3\sqrt{2x^2 - 6x - 11} = x + 33$.
Ans. $x = 6$, or -3, or $\frac{3}{2} \pm \frac{1}{2}\sqrt{273}$.

29. Reduce $21x^4 + 11x^2 - 69 = 321 - (11x^4 + 5x^2)$.
Ans. $x = \pm \frac{1}{2}\sqrt{13}$, or $\pm \frac{1}{2}\sqrt{-15}$.

30. Reduce $\frac{1}{(2x-4)^2} = \frac{1}{8} + \frac{2}{(2x-4)^4}$.

31. Reduce $(x^2 - 4x)^2 = 12x - 3x^2$.

32. Reduce $x + (x^2 - x)^2 = x^2 + 5112$.

PROBLEMS

PRODUCING AFFECTED QUADRATIC EQUATIONS WITH BUT ONE UNKNOWN QUANTITY.

182. Though the numerical negative values obtained in solving the following Problems satisfy the equations formed in accordance with the given conditions, they are practically inadmissible, and are therefore not given in the answers.

1. Divide 40 into two parts such that the sum of their squares shall be 1042.

SOLUTION.

Let $x =$ one part;

then $40 - x =$ other part.

Then, $x^2 + (40 - x)^2 = 1042$

Expanding, $x^2 + 1600 - 80x + x^2 = 1042$

Transposing and uniting, $x^2 - 40x = -279$

Whence, $x = 20 \pm 11 = 31$, or 9

And, $40 - x = 9$, or 31

2. Divide 20 into two parts such that their product will be $99\frac{3}{4}$. Ans. $9\frac{1}{2}$ and $10\frac{1}{2}$.

3. The ages of two brothers are such that the age of the elder plus the square root of the age of the younger is 22 years, and the sum of their ages is 34 years. What is the age of each? Ans. Elder, 18; younger, 16.

NOTE. — The other answers found by reducing the equation, viz. 25 and 9, satisfy the conditions of the equation only upon considering $\sqrt{9} = -3$. To make the problem correspond to these answers, the word "plus" must be changed to "minus."

4. A merchant had two pieces of cloth measuring together 96 yards. The square of the number of yards in the

longer is equal to one hundred times the number of yards in the shorter. How many yards are there in each piece?

Ans. 60 and 36.

5. Find two numbers whose difference is 3, and the sum of whose squares is 117. Ans. 9 and 6.

6. A merchant having sold a piece of cloth that cost him \$42, found that if the price for which he sold it were multiplied by his loss, the product would be equal to the cube of the loss. What was his loss?

NOTE. — If the word "loss" were changed to gain, the other answer, -7, or as it would then become, $+7$, would be correct.

Ans. \$6.

7. Find two numbers whose difference is 5, and product 176. Ans. 11 and 16.

8. There is a square piece of land whose perimeter in rods is 96 less than the number of square rods in the field. What is the length of one side? Ans. 12 rods.

9. Find two numbers whose sum is 8, and the sum of whose cubes is 152.

10. A man bought a number of sheep for \$240, and sold them again for \$6.75 apiece, gaining by the bargain as much as 5 sheep cost him. How many sheep did he buy? Ans. 40.

11. Find two numbers whose difference is 4, and the sum of whose fourth powers is 1312.

NOTE. — Let $x - 2$ and $x + 2$ be the numbers.

Ans. 2 and 6.

12. A man sold a horse for \$312.50, and gained one tenth as much per cent as the horse cost him. How much did the horse cost him? Ans. \$250.

13. The difference of two numbers is 5, and the less minus the square root of the greater is 7. What are the numbers? Ans. 11 and 16.

14. A and B started together for a place 300 miles distant. A arrived at the place 7 hours and 30 minutes before B, who travelled 2 miles less per hour than A. How many miles did each travel per hour?

Ans. A, 10; B, 8 miles.

15. A gentleman distributed among some boys $15; if he had commenced by giving each 10 cents more, 5 of the boys would have received nothing. How many boys were there? Ans. 30.

16. Find two numbers whose sum is a, and product b.

Ans. $\frac{a \pm \sqrt{a^2 - 4b}}{2}$ and $\frac{a \mp \sqrt{a^2 - 4b}}{2}$.

17. A merchant bought a piece of cloth for $45, and sold it for 15 cents more per yard than he paid. Though he gave away 5 yards, he gained $4.50 on the piece. How many yards did he buy, and at what price per yard?

Ans. 60 yards, at 75 cents per yard.

18. A certain number consists of two figures whose sum is 12; and the product of the two figures plus 16 is equal to the number expressed by the figures in inverse order. What is the number? Ans. 84.

19. From a cask containing 60 gallons of pure wine a man drew enough to fill a small keg, and then put into the cask the same quantity of water. Afterward he drew from the cask enough to fill the same keg, and then there were $41\frac{2}{3}$ gallons of pure wine in the cask. How much did the keg hold? Ans. 10 gallons.

20. There is a rectangular piece of land 75 rods long and 65 rods wide, and just within the boundaries there is a ditch of uniform breadth running entirely round the land. The land within the ditch contains 29 acres and 96 square rods. What is the width of the ditch?

Ans. .5 of a rod.

SECTION XX.

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

183. THE DEGREE of any equation is shown by the sum of the indices of the unknown quantities in that term in which this sum is the greatest. Thus,

$4xy - 2x = 7$ is an equation of the second degree,
$bx^2y^2 + xy^2 = a^2c$ " " " fourth "
$5y^4 - 14x = x^2y^3$ " " " fifth "

NOTE. — Before deciding what degree an equation is, it must be cleared of fractions, if the unknown quantities appear both in the denominators and in the numerators or integral terms; and also from negative and fractional exponents.

184. A HOMOGENEOUS EQUATION is one in which the sum of the exponents of the unknown quantities in each term containing unknown quantities is the same. Thus,

$$4x^2 - 4xy + y^2 = 16$$

or
$$x^3 + 3xy^2 + 3x^2y + y^3 = 27$$

or
$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 256$$

is a homogeneous equation.

185. Two quantities enter SYMMETRICALLY into an equation when, whatever their values, they can exchange places without destroying the equation. Thus,

$$x^2 - 2xy + y^2 = 25$$

or
$$x^3 + 3x^2y + 3xy^2 + y^3 = 8$$

or
$$x^2 + 2xy + y^2 + 2x + 2y = 24$$

186. Quadratic equations containing two unknown quantities can generally be solved by the rules already given, if they come under one of the three following cases:—

I. When one of the equations is simple and the other quadratic.

II. When the unknown quantities enter symmetrically into each equation.

III. When each equation is quadratic and homogeneous.

CASE I.

187. When one of the equations is simple and the other quadratic.

1. Given $\begin{cases} 2x + 2y = 22 \\ 3x^2 + y^2 = 111 \end{cases}$, to find x and y.

OPERATION.

$2x + 2y = 22$	(1)	$3x^2 + y^2 = 111$	(2)
$y = 11 - x$	(3)	$3x^2 + 121 - 22x + x^2 = 111$	(4)
		$4x^2 - 22x = -10$	(5)
		$4^2x^2 - () + 11^2 = 121 - 40 = 81$	(6)
		$4x = 11 \pm 9 = 20$, or 2	(7)
$y = 6$, or $10\frac{1}{2}$	(9)	$x = 5$, or $\frac{1}{2}$	(8)

From (1) we obtain (3), or $y = 11 - x$. Substituting this value of y in (2), we obtain (4), an affected quadratic equation, which reduced gives (8); and substituting these values of x in (3), we obtain (9).

In this Case the values of the unknown quantities can generally be found by *substituting in the quadratic equation the value of one unknown quantity found by reducing the simple equation.*

2. Given $\left\{\begin{array}{r} xy = 28 \\ x - y = 3 \end{array}\right\}$, to find x and y.

OPERATION.

$$xy = 28 \quad (1) \qquad\qquad x - y = 3 \quad (2)$$

$$\begin{array}{rll}
x^2 - 2xy + y^2 = & 9 & (3) \\
4xy = & 112 & (4) \\
\hline
x^2 + 2xy + y^2 = & 121 & (5) \\
x + y = & \pm 11 & (6) \\
\hline
2x = & 14, \text{ or } -8 & (7) \\
2y = & 8, \text{ or } -14 & (8) \\
x = & 7, \text{ or } -4 & (9) \\
y = & 4, \text{ or } -7 & (10)
\end{array}$$

Adding four times (1) to the square of (2), we obtain (5); extracting the square root of each member of (5), we obtain (6); adding (2) to (6), we obtain (7); subtracting (2) from (6), we obtain (8); and reducing (7) and (8), we obtain (9) and (10).

NOTE. — Though Example 2 can be solved by the same method as Example 1, the method given is preferable.

By this method find the values of x and y in the following equations: —

3. Given $\left\{\begin{array}{r} x - y = 4 \\ x^2 - y^2 = 32 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 6. \\ y = 2. \end{array}\right.$

4. Given $\left\{\begin{array}{r} x + y = 13 \\ x^2 + y^2 = 85 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 7, \text{ or } 6. \\ y = 6, \text{ or } 7. \end{array}\right.$

5. Given $\left\{\begin{array}{r} xy = 20 \\ 5x + y = 29 \end{array}\right\}$.

6. Given $\left\{\begin{array}{r} xy = 24 \\ 3x - 2y = 10 \end{array}\right\}$.

CASE II.

188. When the unknown quantities enter symmetrically into each equation.

1. Given $\begin{Bmatrix} x + y = 8 \\ x^3 + y^3 = 152 \end{Bmatrix}$, to find x and y.

OPERATION.

$$x + y = 8 \quad (1) \qquad x^3 + y^3 = 152 \quad (2)$$

$$x^2 + 2xy + y^2 = 64 \quad (3)$$

$$x^2 - xy + y^2 = 19 \quad (4)$$

$$3xy = 45 \quad (5)$$

$$xy = 15 \quad (6)$$

$$x^2 - 2xy + y^2 = 4 \quad (7)$$

$$x - y = \pm 2 \quad (8)$$

$$2x = 10, \text{ or } 6 \quad (9)$$

$$2y = 6, \text{ or } 10 \quad (10)$$

$$x = 5, \text{ or } 3 \quad (11)$$

$$y = 3, \text{ or } 5 \quad (12)$$

Squaring (1), we obtain (3); dividing (2) by (1), we obtain (4); subtracting (4) from (3), we obtain (5), from which we obtain (6); subtracting (6) from (4), we obtain (7); extracting the square root of each member of (7), we obtain (8); adding (8) to (1), we obtain (9); subtracting (8) from (1), we obtain (10); and reducing (9) and (10), we obtain (11) and (12).

NOTE 1.—It must not be inferred that x and y are equal to each other in these equations; for when $x = 5$, $y = 3$; and when $x = 3$, $y = 5$. In all the equations under this Case the values of the two unknown quantities are interchangeable.

NOTE 2.—Although $x^3 + y^3 = 152$ is not a quadratic equation, yet as we can combine the two given equations in such a manner as to produce at once a quadratic equation, we introduce it here.

2. Given $\left\{\begin{array}{r} xy = 6 \\ x^2 + y^2 - 2x - 2y = 3 \end{array}\right\}$, to find x and y.

OPERATION.

$$xy = 6 \quad (1) \qquad x^2 + y^2 - 2x - 2y = 3 \quad (2)$$

$$2xy \qquad = 12 \quad (3)$$

$$(x+y)^2 - 2(x+y) = 15 \quad (4)$$

$$(x+y)^2 - (\ ) + 1 = 16 \quad (5)$$

$$x + y = 1 \pm 4 = 5, \text{ or } -3 \quad (6)$$

$$x = 3, \text{ or } 2, \text{ or } \frac{-3 \pm \sqrt{-15}}{2} \quad (7)$$

$$y = 2, \text{ or } 3, \text{ or } \frac{-3 \mp \sqrt{-15}}{2} \quad (8)$$

Adding twice (1) to (2), we obtain (4); completing the square in (4), we obtain (5); extracting the square root of each member of (5), and transposing, we obtain (6); and combining (1) and (6) as the sum and product are combined in the preceding example, we obtain (7) and (8).

In Case II. the process varies as the given equations vary. In general the equations are reduced by *a proper combination of the sum of the squares, or the square of the sum or of the difference, with multiples of the product of the two unknown quantities; and finally, of the sum, with the difference of the two unknown quantities.*

NOTE 3. — When the unknown quantities enter into each equation symmetrically in all respects except their signs, the equations can be reduced by this same method; e. g. $x - y = 7$, and $x^3 - y^3 = 511$. In such equations the values of the unknown quantities are not interchangeable.

NOTE 4. — The signs $\pm$ $\mp$ standing before any quantity taken independently are equivalent to each other; but when one of two quantities is equal to $\pm a$ while the other is equal to $\mp b$, the meaning is that the first is equal to $+a$, when the second is equal to $-b$; and the first to $-a$, when the second is equal to $+b$.

By this method find the values of x and y in the following equations: —

3. Given $\left\{\begin{array}{l} 2x + 2y = 14 \\ 3x^3 + 3y^3 = 273 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 4, \text{ or } 3. \\ y = 3, \text{ or } 4. \end{array}\right.$

4. Given $\left\{\begin{array}{l} x - y = 8 \\ x^3 - y^3 = 728 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 9, \text{ or } -1. \\ y = 1, \text{ or } -9. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x + \sqrt{xy} + y = 14 \\ x^2 + xy + y^2 = 84 \end{array}\right\}$.

NOTE. — Divide the second equation by the first.

6. Given $\left\{\begin{array}{l} x - \sqrt{xy} + y = 7 \\ x^2 + xy + y^2 = 133 \end{array}\right\}$.

CASE III.

189. When each equation is quadratic and homogeneous.

1. Given $\left\{\begin{array}{l} 2xy + y^2 = 5 \\ 3x^2 - xy = 10 \end{array}\right\}$, to find x and y.

OPERATION.

$$2xy + y^2 = 5 \quad (1) \qquad 3x^2 - xy = 10 \quad (2)$$

$$\text{Let } x = vy$$

$$2vy^2 + y^2 = 5 \quad (3) \qquad 3v^2y^2 - vy^2 = 10 \quad (4)$$

$$y^2 = \frac{5}{2v+1} \quad (5) \qquad y^2 = \frac{10}{3v^2 - v} \quad (6)$$

$$\frac{5}{2v+1} = \frac{10}{3v^2 - v} \quad (7)$$

$$15v^2 - 5v = 20v + 10 \quad (8)$$

$$3v^2 - 5v = 2 \quad (9)$$

$$v = 2, \text{ or } -\tfrac{1}{3} \quad (10)$$

$$y^2 = \frac{5}{4+1}, \text{ or } \frac{5}{-\frac{2}{3}+1} \quad (11)$$

$$y = \pm 1, \text{ or } \pm\sqrt{15} \quad (12)$$

$$x = vy = \pm 2, \text{ or } \mp \tfrac{1}{3}\sqrt{15} \quad (13)$$

Substituting vy for x in (1) and (2), we obtain (3) and (4); from (3) and (4) we obtain (5) and (6); putting these two values of y^2 equal to each other, we obtain (7), which reduced gives (10); substituting this value of v in (5), we obtain (11), which reduced gives (12); and substituting in $x = vy$ the values of v and y from (10) and (12), we obtain (13).

Examples under Case III. can generally be reduced best by *substituting for one of the unknown quantities the product of the other by some unknown quantity, and then finding the value of this third unknown quantity.* When the value of this third quantity becomes known, the values of the given unknown quantities can be readily found by substitution.

NOTE. — Whenever, as in the example above, the square root is taken twice, each unknown quantity has four values; but these values must be taken in the same order, i. e. in the example above, when $y = +1$, $x = +2$; when $y = -1$, $x = -2$; when $y = +\sqrt{15}$, $x = -\frac{1}{3}\sqrt{15}$; and when $y = -\sqrt{15}$, $x = +\frac{1}{3}\sqrt{15}$.

By this method find the values of x and y in the following equations: —

2. Given $\left\{\begin{array}{c} x^2 - xy = 14 \\ 3xy - 2y^2 = 55 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = \pm 7, \text{ or } \pm 4\sqrt{-\frac{1}{2}}. \\ y = \pm 5, \text{ or } \pm 11\sqrt{-\frac{1}{2}}. \end{array}\right.$

3. Given $\left\{\begin{array}{c} x^2 + 3xy = 27 \\ 2xy + y^2 = 16 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = \pm 3, \text{ or } \pm 9\sqrt{-\frac{1}{5}}. \\ y = \pm 2, \text{ or } \mp 8\sqrt{-\frac{1}{5}}. \end{array}\right.$

4. Given $\left\{\begin{array}{c} x^2 + 4xy = 14 - 2y^2 \\ xy - 3y^2 = 3 - x^2 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x = \pm 2, \text{ or } \pm 24\sqrt{-\frac{1}{17}}. \\ y = \pm 1, \text{ or } \mp 11\sqrt{-\frac{1}{17}}. \end{array}\right.$

5. Given $\left\{\begin{array}{c} 32 - 3xy = 2x^2 - y^2 \\ x^2 - 3 = y^2 + 2 \end{array}\right\}$.

190. Find the values of x and y in the following

EXAMPLES.

NOTE.—Some of the examples given below belong at the same time to two Cases. Thus in Example 1 both the equations are symmetrical, and both are quadratic and homogeneous, and therefore it belongs both to Case II. and Case III. Example 3 belongs both to Case I. and Case II.

1. Given $\left\{\begin{array}{r} xy = 20 \\ x^2 + y^2 = 41 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = \pm 5, \text{ or } \pm 4. \\ y = \pm 4, \text{ or } \pm 5. \end{array}\right.$

2. Given $\left\{\begin{array}{r} xy = 6 \\ x^2 + 7xy = 55 - y^2 \end{array}\right\}$.

3. Given $\left\{\begin{array}{l} x + y = 7 \\ x^2 + y = 32 - (x + y^2) \end{array}\right\}$.

   Ans. $\left\{\begin{array}{l} x = 4, \text{ or } 3. \\ y = 3, \text{ or } 4. \end{array}\right.$

4. Given $\left\{\begin{array}{r} xy = 12 \\ x^2 + x = 32 - y - y^2 \end{array}\right\}$.

5. Given $\left\{\begin{array}{r} \dfrac{x^2 - y^2}{x - y} = 10 \\ 3xy - 7 = 56 \end{array}\right\}$. Ans. $\left\{\begin{array}{l} x = 7, \text{ or } 3. \\ y = 3, \text{ or } 7. \end{array}\right.$

6. Given $\left\{\begin{array}{r} x - y = 2 \\ x^2y^2 + 2xy = 1295 \end{array}\right\}$.

NOTE.—Considering xy a single quantity, find its value in the second equation.

7. Given $\left\{\begin{array}{r} x^2y - xy^2 = 30 \\ x^3 - y^3 = 98 \end{array}\right\}$.

NOTE.—Subtract from the second equation three times the first, and extract the cube root of each member of the resulting equation.

8. Given $\left\{\begin{array}{r} 2x^2y + 2xy^2 = 168 \\ x^3 + y^3 = 91 \end{array}\right\}$.

   Ans. $\left\{\begin{array}{l} x = 4, \text{ or } 3. \\ y = 3, \text{ or } 4. \end{array}\right.$

9. Given $\left\{\begin{matrix} 3x^2y - 3xy^2 = 18 \\ 2x^3y^2 - 2x^2y^3 = 36 \end{matrix}\right\}$.

10. Given $\left\{\begin{matrix} x^3 - y^3 = \dfrac{13\,(x-y)^3}{3} \\ xy = 10 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = \pm\, 5. \\ y = \pm\, 2. \end{matrix}\right.$

11. Given $\left\{\begin{matrix} 7x^2 - 2xy = 88 \\ 3y^2 + 4xy = 75 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = \pm\, 4, \text{ or } \pm\ 66\sqrt{\frac{1}{609}}. \\ y = \pm\, 3, \text{ or } \mp\, 175\sqrt{\frac{1}{609}}. \end{matrix}\right.$

12. Given $\left\{\begin{matrix} x^2 - 2x + 2y = 30 - y^2 \\ 4xy = 60 \end{matrix}\right\}$.

13. Given $\left\{\begin{matrix} x^{\frac{2}{3}}y^{\frac{3}{2}} = 4y^2 \\ 6x^{\frac{1}{3}} - 2y^{\frac{1}{2}} = 10 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = 1000, \text{ or } 8. \\ y = 625, \text{ or } 1. \end{matrix}\right.$

14. Given $\left\{\begin{matrix} 3xy = 18 \\ x^4 - y^4 = 65 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = \pm\, 3, \text{ or } \pm\, 2\sqrt[4]{-1}. \\ y = \pm\, 2, \text{ or } \pm\, 3\sqrt[4]{-1}. \end{matrix}\right.$

15. Given $\left\{\begin{matrix} 3\,(x - y) = 3\,(\sqrt{x} + \sqrt{y}) \\ xy = 36 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = \dfrac{\pm\sqrt{-23} - 11}{2}, \text{ or } 9, \text{ or } 4. \\ y = \dfrac{\mp\sqrt{-23} - 11}{2}, \text{ or } 4, \text{ or } 9. \end{matrix}\right.$

16. Given $\left\{\begin{matrix} \sqrt{x} - \sqrt{y} = \dfrac{x-y}{9} \\ x + y = 41 \end{matrix}\right\}$.

17. Given $\left\{\begin{matrix} x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1 \\ x - y = 19 \end{matrix}\right\}$.

18. Given $\left\{\begin{array}{r} \sqrt[3]{x}+\sqrt[3]{y}=5 \\ x+y=65 \end{array}\right\}$.

19. Given $\left\{\begin{array}{l} x^{-2}-y^{-2}=\frac{5}{36} \\ x^{-1}-y^{-1}=\frac{1}{6} \end{array}\right\}$.

20. Given $\left\{\begin{array}{l} x^3+2x^2y+2xy^2+y^3=95 \\ x^3-x^2y-xy^2+y^3=5 \end{array}\right\}$.

21. Given $\left\{\begin{array}{r} xy=8 \\ x^4+y^4=272 \end{array}\right\}$.

22. Given $\left\{\begin{array}{r} x+y=6 \\ x^4+y^4=626 \end{array}\right\}$.

PROBLEMS

PRODUCING QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

191. Though the numerical negative values obtained in solving the following Problems satisfy the equations formed in accordance with the given conditions, they are practically inadmissible, and, except in Example 4, are not given in the answers.

1. The sum of the squares of two numbers plus the sum of the two numbers is 98; and the product of the two numbers is 42. What are the numbers?

Ans. 7 and 6.

2. If a certain number is divided by the product of its figures the quotient will be 3; and if 18 is added to the number, the order of the figures will be inverted. What is the number?

Ans. 24.

3. A certain number consists of two figures whose product is 21; and if 22 is subtracted from the number,

and the sum of the squares of its figures added to the remainder, the order of the figures will be inverted. What is the number? Ans. 37.

4. Find two numbers such that their sum, their product, and the difference of their squares shall be equal to one another. Ans. $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$ and $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

5. There are two pieces of cloth of different lengths; and the sum of the squares of the number of yards in each is 145; and one half the product of their lengths plus the square of the length of the shorter is 100. What is the length of each?

Ans. Shorter, 8; longer, 9 yards.

6. Find two numbers such that the greater shall be to the less as the less is to $2\frac{2}{7}$, and the difference of their squares shall be 33.

7. The area of a rectangular field is 1575 square rods; and if the length and breadth were each lessened 5 rods, its area would be 1200 square rods. What are the length and breadth?

8. Find two numbers such that their sum shall be to 6 as 9 is to the greater, and the sum of their squares shall be 45. Ans. $9\sqrt{\frac{1}{2}}$ and $3\sqrt{\frac{1}{2}}$, or 6 and 3.

9. The fore wheels of a carriage make 2 revolutions more than the hind wheels in going 90 yards; but if the circumference of each wheel is increased 3 feet, the carriage must pass over 132 yards in order that the fore wheels may make 2 revolutions more than the hind wheels. What is the circumference of each wheel?

Ans. Fore wheels, $13\frac{1}{2}$ feet; hind wheels, 15 feet.

10. Find two numbers such that five times the square of the greater plus three times their product shall be 104, and three times the square of the less minus their product shall be 4.

SECTION XXI.

RATIO AND PROPORTION.

192. RATIO is the relation of one quantity to another of the same kind; or, it is the quotient which arises from dividing one quantity by another of the same kind.

Ratio is indicated by writing the two quantities after one another with two dots between, or by expressing the division in the form of a fraction. Thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$; read, a is to b, or a divided by b.

193. The TERMS of a ratio are the quantities compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, and the other the *consequent*; and the two terms together are called a *couplet*.

194. An INVERSE, or RECIPROCAL RATIO, of any two quantities is the ratio of their *reciprocals*. Thus, the *direct* ratio of a to b is $a : b$, i. e. $\frac{a}{b}$; and the *inverse* ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, i. e. $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b : a$.

195. PROPORTION is an equality of ratios. Four quantities are proportional when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=) or by four dots (: :).

Thus, $a : b = c : d$, or $a : b : : c : d$, or $\frac{a}{b} = \frac{c}{d}$; read, a to b equals c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

196. In a proportion the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proportion. The first and fourth terms are called the *extremes*, and the second and third the *means*.

197. When three quantities are in proportion, e. g. $a : b = b : c$, the second is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

198. A proportion is transformed by ALTERNATION when antecedent is compared with antecedent, and consequent with consequent.

199. A proportion is transformed by INVERSION when the antecedents are made consequents, and the consequents antecedents.

200. A proportion is transformed by COMPOSITION when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

201. A proportion is transformed by DIVISION when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

THEOREM I.

202. *In a proportion the product of the extremes is equal to the product of the means.*

Let $$a : b = c : d$$

i. e. $$\frac{a}{b} = \frac{c}{d}$$

Clearing of fractions, $$ad = bc$$

THEOREM II.

203. *If the product of two quantities is equal to the product of two others, the factors of either product may be made the extremes, and the factors of the other the means of a proportion.*

Let $ad = bc$

Dividing by bd, $\frac{a}{b} = \frac{c}{d}$.

i. e. $a : b = c : d$

THEOREM III.

204. *If four quantities are in proportion, they will be in proportion by alternation.*

Let $a : b = c : d$

By Theorem I. $ad = bc$

By Theorem II. $a : c = b : d$

THEOREM IV.

205. *If four quantities are in proportion, they will be in proportion by inversion.*

Let $a : b = c : d$

By Theorem I. $ad = bc$

By Theorem II. $b : a = d : c$

THEOREM V.

206. *If three quantities are in proportion, the product of the extremes is equal to the square of the mean.*

Let $a : b = b : c$

By Theorem I. $ac = b^2$

THEOREM VI.

207. *If four quantities are in proportion, they will be in proportion by composition.*

Let $$a : b = c : d$$

i. e. $$\frac{a}{b} = \frac{c}{d}$$

Adding 1 to each member, $$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

or $$\frac{a+b}{b} = \frac{c+d}{d}$$

i. e. $$a + b : b = c + d : d$$

THEOREM VII.

208. *If four quantities are in proportion, they will be in proportion by division.*

Let $$a : b = c : d$$

i. e. $$\frac{a}{b} = \frac{c}{d}$$

Subtracting 1 from each member, $$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

or $$\frac{a-b}{b} = \frac{c-d}{d}$$

i. e. $$a - b : b = c - d : d$$

THEOREM VIII.

209. *Two ratios respectively equal to a third are equal to each other.*

Let $$a : b = m : n \text{ and } c : d = m : n$$

i. e. $$\frac{a}{b} = \frac{m}{n} \quad \text{and} \quad \frac{c}{d} = \frac{m}{n}$$

Hence (Art. 13, Ax. 8), $$\frac{a}{b} = \frac{c}{d}$$

i. e. $$a : b = c : d$$

THEOREM IX.

210. *If four quantities are in proportion, the sum and difference of the terms of each couplet will be in proportion.*

Let	$a : b = c : d$
By Theorem VI.	$a + b : b = c + d : d$ (1)
and by Theorem VII.	$a - b : b = c - d : d$ (2)
From (1), by Theorem III.	$a + b : c + d = b : d$
From (2), by Theorem III.	$a - b : c - d = b : d$
By Theorem VIII.	$a + b : c + d = a - b : c - d$
Hence, by Theorem III.	$a + b : a - b = c + d : c - d$

THEOREM X.

211. *Equimultiples of two quantities have the same ratio as the quantities themselves.*

For by Art. 83, $$\frac{a}{b} = \frac{ma}{mb}$$

i. e. $$a : b = ma : mb$$

Cor. It follows that either couplet of a proportion may be multiplied or divided by any quantity, and the resulting quantities will be in proportion. And since by Theorem III. if $a : b = ma : mb$, $a : ma = b : mb$, or $ma : a = mb : b$, it follows that both consequents, or both antecedents, may be multiplied or divided by any quantity, and the resulting quantities will be in proportion.

THEOREM XI

212. *If four quantities are in proportion, like powers or like roots of these quantities will be in proportion.*

Let $$a : b = c : d$$

i. e. $$\frac{a}{b} = \frac{c}{d}$$

Hence, $$\frac{a^n}{b^n} = \frac{c^n}{d^n}$$

i. e. $$a^n : b^n = c^n : d^n$$

Since n may be either integral or fractional, the theorem is proved.

THEOREM XII.

213. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b = c : d = e : f$

Now $ab = ab$ (1)

and by Theorem I. $ad = bc$ (2)

and also $af = be$ (3)

Adding (1), (2), (3), $a(b + d + f) = b(a + c + e)$

Hence, by Theorem II. $a : b = a + c + e : b + d + f$

THEOREM XIII.

214. *If there are two sets of quantities in proportion, their products, or quotients, term by term, will be in proportion.*

Let $a : b = c : d$

and $e : f = g : h$

By Theorem I. $ad = bc$ (1)

and $eh = fg$ (2)

Multiplying (1) by (2), $adeh = bcfg$; (3)

Dividing (1) by (2), $\frac{ad}{eh} = \frac{bc}{fg}$. (4)

From (3), by Theorem II. $ae : bf = cg : dh$

and from (4), $\frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}$.

PROBLEMS IN PROPORTION.

215. By means of the principles just demonstrated, a proportion may often be very much simplified before making the product of the means equal to the product of the extremes; and a proportion which could not otherwise be reduced by the ordinary rules of Algebra may often be so simplified as to produce a simple equation.

1. The cube of the smaller of two numbers multiplied by four times the greater is 96; and the sum of their cubes is to the difference of their cubes as 210 : 114. What are the numbers?

SOLUTION.

Let $x =$ the greater and $y =$ the less.

Then $4xy^3 = 96$ (1) $x^3 + y^3 : x^3 - y^3 = 210 : 114$ (2)

From (2), by Theo. X., Cor. $x^3 + y^3 : x^3 - y^3 = 35 : 19$

By Theorem IX. $2x^3 : 2y^3 = 54 : 16$

By Theorem X., Cor. $x^3 : y^3 = 27 : 8$

By Theorem XI. $x : y = 3 : 2$

By Theorem I. $2x = 3y$ (3)

From (1) and (3) we find $x = 3$ and $y = 2$.

2. The product of two numbers is 78; and the difference of their cubes is to the cube of their difference as 283 : 49. What are the numbers?

SOLUTION.

Let $x =$ the greater and $y =$ the less.

Then $xy = 78$ (1) $x^3 - y^3 : x^3 - 3x^2y + 3xy^2 - y^3 = 283 : 49$ (2)

From (2), by division, $3x^2y - 3xy^2 : (x - y)^3 = 234 : 49$

Dividing 1st couplet by $x - y$, $3xy : (x - y)^2 = 234 : 49$

Dividing antecedents by 3, $xy : (x - y)^2 = 78 : 49$

Substituting the value of xy, $78 : (x - y)^2 = 78 : 49$

Dividing antecedents by 78, $1 : (x - y)^2 = 1 : 49$

Extracting the square root, $1 : x - y = 1 : 7$

Whence, $x - y = 7$ (3)

From (1) and (3) we find $x = 13$ and $y = 6$.

3. The sum of the cubes of two numbers is to the cube of their sum as 13 : 25; and 4 is a mean proportional between them. What are the numbers?

4 The difference of two numbers is 10; and their product is to the sum of their squares as 6 : 37. What are the numbers?

SOLUTION.

Let $x =$ the greater and $y =$ the less.

Then $x - y = 10$ (1) $xy : x^2 + y^2 = 6 : 37$ (2)

From (2), by Theorem X., Cor. $2xy : x^2 + y^2 = 12 : 37$

By Theorem IX. $x^2 + 2xy + y^2 : x^2 - 2xy + y^2 = 49 : 25$

By Theorem XI. $x + y : x - y = 7 : 5$

By Theorem IX. $2x : 2y = 12 : 2$

By Theorem X., Cor. $x : y = 6 : 1$

By Theorem I. $x = 6y$ (3)

From (1) and (3) we find $x = 12$ and $y = 2$.

5. The product of two numbers is 136; and the difference of their squares is to the square of their difference as 25 : 9. What are the numbers? Ans. 8 and 17.

6. As two boys were talking of their ages, they discovered that the product of the numbers representing their ages in years was 320, and the sum of the cubes of these same numbers was to the cube of their sum as 7 : 27. What was the age of each?

Ans. Younger, 16; elder, 20 years.

7. As two companies of soldiers were returning from the war, it was found that the number in the first multiplied by that in the second was 486, and the sum of the squares of their numbers was to the square of the sum as 13 : 25. How many soldiers were there in each company?

Ans. In 1st, 27; in 2d, 18.

8. The difference of two numbers is to the less as 100 is to the greater; and the same difference is to the greater as 4 is to the less. What are the numbers?

NOTE. — Multiply the two proportions together. (Theorem XIII.)

SECTION XXII.

PROGRESSION.

216. A PROGRESSION is a series in which the terms increase or decrease according to some fixed law.

217. The TERMS of a series are the several quantities, whether simple or compound, that form the series. The first and last terms are called the *extremes*, and the others the *means*.

ARITHMETICAL PROGRESSION.

218. An ARITHMETICAL PROGRESSION is a series in which each term, except the first, is derived from the preceding by the addition of a constant quantity called the *common difference.*

219. When the common difference is positive, the series is called an *ascending* series, or an *ascending* progression; when the common difference is negative, a *descending* series. Thus,

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \&c.$$

is an ascending arithmetical series in which the common difference is d; and

$$a,\ a - d,\ a - 2d,\ a - 3d,\ \&c.$$

is a descending arithmetical series in which the common difference is $-d$.

220. In Arithmetical Progression there are five elements, any three of which being given, the other two can be found: —

1. The first term.
2. The last term.

3. The common difference.
4. The number of terms.
5. The sum of all the terms.

221. Twenty cases may arise in Arithmetical Progression. In discussing this subject we shall let

$a =$ the first term,

$l =$ the last term,

$d =$ the common difference,

$n =$ the number of terms,

$S =$ the sum of all the terms.

CASE I.

222. The first term, common difference, and number of terms given, to find the last term.

In this Case a, d, and n are given, and l is required. The successive terms of the series are

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ \&c.;$$

that is, the coefficient of d in each term is one less than the number of that term, counting from the left; therefore the last or nth term in the series is

$$a + (n - 1)\,d$$

or $$l = a + (n - 1)\,d$$

in which the series is ascending or descending according as d is positive or negative. Hence,

RULE.

To the first term add the product formed by multiplying the common difference by the number of terms less one.

1. Given $a = 4$, $d = 2$, and $n = 9$, to find l.
$l = a + (n - 1)\,d = 4 + (9 - 1)\,2 = 20$, Ans.

2. Given $a = 7$, $d = 3$, and $n = 19$, to find l.
Ans. $l = 61$.

3. Given $a = 29$, $d = -2$, and $n = 14$, to find l.
Ans. $l = 3$.

4. Given $a = 40$, $d = 10$, and $n = 100$, to find l.

5. Given $a = 1$, $d = \frac{1}{2}$, and $n = 17$, to find l.

6. Given $a = \frac{7}{8}$, $d = -\frac{1}{16}$, and $n = 13$, to find l.

7. Given $a = .01$, $d = -.001$, and $n = 10$, to find l.

CASE II.

223. The extremes and the number of terms given, to find the sum of the series.

In this Case a, l, and n are given, and S is required.

Now $$S = a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots + l$$

or, inverting the series, $$S = l + (l - d) + (l - 2d) + (l - 3d) + \ldots\ldots + a$$

Adding these together, $$2S = (a + l) + (a + l) + (a + l) + (a + l) + \ldots\ldots + (a + l)$$

And since $(a + l)$ is to be taken as many times as there are terms,

hence $$2S = n(a + l)$$

or $$S = \frac{n}{2}(a + l).$$ Hence,

RULE.

Find one half the product of the sum of the extremes and the number of terms.

NOTE. — If in place of the last term the common difference is given, the last term must first be found by the Rule in Case I.

1. Given $a = 3$, $l = 141$, and $n = 26$, to find S.

$$S = \frac{n}{2}(a + l) = \frac{26}{2}(3 + 141) = 1872, \text{ Ans.}$$

2. Given $a = \frac{1}{5}$, $l = 25$, and $n = 63$, to find S.
Ans. $S = 793\frac{4}{5}$.

3. Given $a = 4$, $d = 2$, and $n = 24$, to find S.
Ans. $S = 648$.

4. Given $a = -3$, $d = 2$, and $n = 4$, to find S.
Ans. $S = 0$.

5. Given $a = \frac{1}{3}$, $d = -\frac{1}{3}$, and $n = 3$, to find S.

6. Given $a = .07$, $l = .77$, and $n = 11$, to find S.

7. Given $a = -4\frac{1}{2}$, $d = \frac{1}{7}$, and $n = 25$, to find S.

CASE III.

224. The extremes and number of terms given, to find the common difference.

In this case a, l, and n are given, and d is required.

From Case I. we have $\quad l = a + (n - 1)\,d$

Transposing and reducing, $\quad d = \dfrac{l - a}{n - 1}$. Hence,

RULE.

Divide the last term minus the first term by the number of terms less one, and the quotient will be the common difference.

1. Given $a = 5$, $l = 47$, and $n = 7$, to find d.

$$d = \frac{l - a}{n - 1} = \frac{47 - 5}{7 - 1} = 7, \text{ Ans.}$$

2. Given $a = 27$, $l = 148$, and $n = 12$, to find d.

Ans. $d = 11$.

3. Given $a = 41$, $l = 3$, and $n = 20$, to find d.

Ans. $d = -2$.

4. Given $a = \frac{17}{28}$, $l = \frac{1}{14}$, and $n = 6$, to find d.

Ans. $d = -\frac{3}{28}$.

5. Given $a = .09$, $l = .9$, and $n = 10$, to find d.

NOTE. — This rule enables us to insert any number of arithmetical means between two given quantities; for the number of terms is two greater than the number of means. Hence, if $m =$ the number of means, $m + 2 = n$, or $m + 1 = n - 1$, and $d = \dfrac{l - a}{m + 1}$. Having found the common difference, the means are found by adding the common difference once, twice, &c., to the first term.

6. Find 6 arithmetical means between 3 and 38.
Ans. 8, 13, 18, 23, 28, 33.

7. Find 3 arithmetical means between 3 and 27.

8. Find 5 arithmetical means between 1 and 37.

9. Find 7 arithmetical means between 2 and 26.

NOTE. — When $m = 1$, the formula becomes

$$d = \frac{l - a}{2}$$

Adding a to each member,

$$a + d = \frac{l - a}{2} + a = \frac{l + a}{2}$$

But $a + d$ is the second term of a series whose first term is a and common difference d, or the arithmetical mean of the series a, $a + d$, $a + 2d$. Hence, *the arithmetical mean between two quantities is one half of their sum.*

10. Find the arithmetical mean between 7 and 17.
Ans. 12.

11. Find the arithmetical mean between $\frac{1}{8}$ and $\frac{7}{8}$.

12. Find the arithmetical mean between 4 and -4.

225. From the formulas established in Arts. 222 and 223, viz.

$$l = a + (n - 1) d \qquad (1)$$

$$S = \frac{n}{2}(a + l) \qquad (2)$$

can be derived formulas for all the Cases in Arithmetical Progression.

From (1) we can obtain the value of any one of the four quantities, l, a, n, or d, when the other three are given; and from (2) the value of any one of the four quantities, S, n, a, or l, when the other three are given. Formulas for the remaining twelve Cases which may arise are derived by combining the two formulas (1) and (2), so as to eliminate that one of the two unknown quantities whose value is not sought.

1. Find the formula for the value of n, when a, d, and S are given.

OPERATION.

$$l = a + (n-1)d \qquad (1) \qquad\qquad S = \frac{n}{2}(a+l) \qquad (2)$$

$$ln = an + dn^2 - dn \qquad (3) \qquad 2S - an = ln \qquad (4)$$

$$an + dn^2 - dn = 2S - an \qquad (5)$$

$$n^2 - \left(\frac{d-2a}{d}\right)n = \frac{2S}{d} \qquad (6)$$

$$n^2 - (\ ) + \frac{(d-2a)^2}{4d^2} = \frac{(d-2a)^2}{4d^2} + \frac{2S}{d} \qquad (7)$$

$$n = \frac{d - 2a \pm \sqrt{(d-2a)^2 + 8dS}}{2d} \qquad (8)$$

To obtain the formula required in this example, l must be eliminated from (1) and (2). From (1) and (2) we obtain (3) and (4). Placing these two values of ln equal to each other, we form (5), which reduced gives (8), or the value of n in known quantities.

2. Find the formula for the value of n, when d, l, and S are given. Ans. $n = \dfrac{2l + d \pm \sqrt{(2l+d)^2 - 8dS}}{2d}$.

3. Find the formula for the value of S, when a, d, and n are given. Ans. $S = \frac{1}{2}n\,[2a + (n-1)d]$.

4. Find the formula for the value of S, when a, d, and l are given. Ans. $S = \dfrac{(l+a)(l-a+d)}{2d}$.

5. Find the formula for the value of S, when d, n, and l are given. Ans. $S = \frac{1}{2}n\,[2l - (n-1)d]$.

6. Find the formula for the value of l, when a, d, and S are given. Ans. $l = -\dfrac{d}{2} \pm \sqrt{\left(a - \dfrac{d}{2}\right)^2 + 2dS}$.

7. Find the formula for the value of l, when d, n, and S are given. Ans. $l = \dfrac{2S + n(n-1)d}{2n}$.

8. Find the formula for the value of d, when a, n, and S are given.

Ans. $d = \frac{2S - 2an}{n(n-1)}$.

9. Find the formula for the value of d, when a, l, and S are given.

Ans. $d = \frac{(l+a)(l-a)}{2S - (l+a)}$.

10. Find the formula for the value of d, when n, l, and S are given.

Ans. $d = \frac{2(nl - S)}{n(n-1)}$.

11. Find the formula for the value of a, when d, n, and S are given.

Ans. $a = \frac{2S - n(n-1)d}{2n}$.

12. Find the formula for the value of a, when d, l, and S are given.

Ans. $a = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2} + l\right)^2 - 2dS}$.

226. To find any one of the five elements when three others are given.

RULE.

Substitute the given values in that formula whose first member is the required term, and whose second contains the three given terms.

1. Given $d = 2$, $l = 21$, and $S = 120$, to find a.

OPERATION.

$$a = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2} + l\right)^2 - 2dS} \qquad (1)$$

$$a = \frac{2}{2} \pm \sqrt{\left(\frac{2}{2} + 21\right)^2 - 2.2.120} \qquad (2)$$

$$a = 3, \text{ or } -1 \qquad (3)$$

In Example 12, Art. 225, we find (1), the required formula; substituting the given values of d, l, and S, we obtain (2), which reduced gives (3), or $a = 3$, or -1.

NOTE.—If $a = 3$, $n = 10$; but if $a = -1$, $n = 12$.

2. Given $d = \frac{1}{3}$, $l = 27$, and $S = 392$, to find n.
Ans. $n = 147$, or 16.

NOTE. — When $n = 147$, $a = -21\frac{2}{3}$; but when $n = 16$, $a = 22$.

3. Given $d = 7$, $n = 6$, and $S = 135$, to find l.
Ans. $l = 40$.

4. Given $d = -2$, $n = 6$, and $a = 5$, to find S.
Ans. $S = 0$.

5. Given $a = \frac{1}{5}$, $l = 15$, and $S = 288\frac{4}{5}$, to find d.
Ans. $d = \frac{2}{5}$.

6. Find the 100th term of the series 3, 10, 17, &c.
Ans. 696.

7. Find the sum of 100 terms of the series 3, 10, 17, &c.
Ans. 34950.

8. Find the common difference and sum of the series whose first term is 25, last term 95, and number of terms 15. Ans. $d = 5$; $S = 900$.

9. Find the sum of the natural series of numbers from 1 to 100, inclusive.

10. Find the sum of 10 of the odd numbers 1, 3, 5, &c.

11. Find the sum of 10 of the even numbers 2, 4, 6, &c.

12. How many strokes does a clock strike in 12 hours?

13. If 100 trees stand in a straight line 10 feet from one another, how far must a person, starting from the first tree and returning to it each time, travel to go to every tree? Ans. $18\frac{3}{4}$ miles.

14. If a person should save a cent the first day, two cents the second, three the third, and so on, how much would he save in 365 days? Ans. \$667.95.

15. If a person should save \$25 a year and put this sum at simple interest at 5 per cent at the end of each year, to how much would it amount at the end of 25 years?

PROBLEMS
TO WHICH THE FORMULAS DO NOT DIRECTLY APPLY.

227. Sometimes in examples in progression the terms are not directly given, but are implied in the conditions of the problem. In this case the formulas cannot be directly used, but the terms can be represented by unknown quantities, and equations formed according to the given conditions.

228. If $x =$ first term and $y =$ the common difference; then

$$x,\ x+y,\ x+2y,\ x+3y, \text{ \&c.}$$

will represent the series.

It will often be found more convenient when the number of terms is odd to represent the middle term by x and the common difference by y; then the series for three terms will be

$$x-y,\ x,\ x+y;$$

and for five terms,

$$x-2y,\ x-y,\ x,\ x+y,\ x+2y;$$

and when the number of terms is even, to represent the two middle terms by $x-y$ and $x+y$, and the common difference by $2y$; then the series for four terms is

$$x-3y,\quad x-y,\quad x+y,\quad x+3y.$$

The advantage of this latter method is, that the sum of the series, or the sum or difference of any two terms equally distant from the mean, or means, will contain but one unknown quantity.

1. The sum of three numbers in arithmetical progression is 15, and the sum of their squares is 83. What are the numbers?

Let $x =$ the mean term and $y =$ the common difference; then the series will be $x - y$, x, and $x + y$.

By the conditions, $$3x = 15 \quad (1)$$

and $$3x^2 + 2y^2 = 83 \quad (2)$$

Ans. 3, 5, 7.

2. The sum of four numbers in arithmetical progression is 44, and the sum of the cubes of the two means is 2926. Ans. 5, 9, 13, 17.

3. Find seven numbers in arithmetical progression such that the sum of the first and fifth shall be 10, and the difference of the squares of the second and fourth 40.

4. There are four numbers in arithmetical progression; the product of the first and third is 20; and the product of the second and fourth 84. What are the numbers?

Ans. 2, 6, 10, 14.

5. The sum of four numbers in arithmetical progression is 32; and their product 3465. What are the numbers?

Ans. 5, 7, 9, 11.

6. The sum of the squares of the extremes of four numbers in arithmetical progression is 461; and the sum of the squares of the means 425. What are the numbers?

Ans. 10, 13, 16, 19.

7. A certain number consists of three figures which are in arithmetical progression; if the number is divided by the sum of its figures, the quotient will be 15; and if 396 is added to the number, the order of the figures will be inverted. What is the number? Ans. 135.

8. Find four numbers in arithmetical progression such that the sum of the squares of the first and third shall be 104, and of the second and fourth 232.

9. Find four numbers in arithmetical progression such that the sum of the squares of the first and second shall be 29, and of the third and fourth 185.

SECTION XXIII.

GEOMETRICAL PROGRESSION.

229. A GEOMETRICAL PROGRESSION is a series in which each term, except the first, is derived by multiplying the preceding term by a constant quantity called the *ratio.*

230. If the first term is positive, when the ratio is a positive integral quantity, the series is called an *ascending* series, and when the ratio is a positive proper fraction, a *descending* series; but if the first term is negative, the series is ascending when the ratio is a positive proper fraction, and descending when the ratio is a positive integral quantity. Thus,

$$\left.\begin{array}{rrrrl} 2, & 6, & 18, & 54, & \&c. \\ -54, & -18, & -6, & -2, & \&c. \end{array}\right\} \text{are ascending series;}$$

$$\left.\begin{array}{rrrrl} 64, & 32, & 16, & 8, & \&c. \\ -8, & -16, & -32, & -64, & \&c. \end{array}\right\} \text{are descending series.}$$

If the ratio is negative, the terms of the progression are alternately positive and negative. Thus, if the ratio is -2 and the first term 3, the series will be

$$3, \ -6, \ +12, \ -24, \ +48, \ \&c.;$$

but if the first term is -3,

$$-3, \ +6, \ -12, \ +24, \ -48, \ \&c.$$

The positive terms of these two series constitute an ascending progression whose ratio is the square of the given ratio; and the negative terms a descending progression having the same ratio.

231. In Geometrical Progression there are five elements, any three of which being given, the other two can be found. These elements are the same as in Arithmetical Progression, except that in place of the common difference we have *the ratio.*

232. Twenty cases may arise in Geometrical Progression. In discussing these cases we shall preserve the same notation as in Arithmetical Progression, except that instead of $d =$ the common difference we shall use $r =$ the ratio.

CASE I.

233. The first term, ratio, and number of terms given, to find the last term.

In this Case a, r, and n are given, and l required.

The successive terms of the series are

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \&c.$$

That is, each term is the product of the first term and that power of the ratio which is one less than the number of that term counting from the left; therefore the last or nth term in the series is

$$ar^{n-1}$$

or $$l = ar^{n-1}. \text{ Hence,}$$

RULE.

Multiply the first term by that power of the ratio whose index is one less than the number of terms.

1. Given $a = 7$, $r = 3$, and $n = 5$, to find l.
$l = ar^{n-1} = 7 \times 3^4 = 567$, Ans.

2. Given $a = 3$, $r = 2$, and $n = 9$, to find l.
Ans. $l = 768$.

3. Given $a = 64$, $r = \frac{1}{2}$, and $n = 10$, to find l.
Ans. $l = \frac{1}{8}$.

4. Given $a = -7$, $r = -4$, and $n = 3$, to find l.
Ans. $l = -112$.

5. Given $a = -\frac{1}{9}$, $r = \frac{1}{3}$, and $n = 5$, to find l.
Ans. $l = -\frac{1}{729}$.

6. Given $a = 5$, $r = -\frac{1}{2}$, and $n = 10$, to find l.

7. Given $a = -\frac{1}{3}$, $r = \frac{1}{3}$, and $n = 8$, to find l.

8. Given $a = -10$, $r = -2$, and $n = 6$, to find l.

CASE II.

234. The extremes, and the ratio given, to find the sum of the series.

In this Case a, l, and r are given, and S is required.

Now $\qquad S = a + ar + ar^2 + ar^3 + \ldots\ldots + l \qquad (1)$

Multiplying (1) by r, $\qquad rS = ar + ar^2 + ar^3 + \ldots\ldots + l + lr \quad (2)$

Subtracting (1) from (2), $\quad rS - S = lr - a$

Whence, $\qquad S = \dfrac{lr - a}{r - 1}$. Hence,

RULE.

Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.

1. Given $a = 2$, $l = 20000$, and $r = 10$, to find S.

$$S = \frac{lr - a}{r - 1} = \frac{20000 \times 10 - 2}{10 - 1} = 22222, \text{ Ans.}$$

2. Given $a = 7$, $l = 45927$, and $r = 3$, to find S.
Ans. $S = 68887$.

3. Given $a = -5$, $l = -405$, and $r = 3$, to find S.

4. Given $a = -\frac{1}{1715}$, $l = \frac{1}{5}$, and $r = -7$, to find S.
Ans. $S = \frac{60}{343}$.

CASE III.

235. The first term, ratio, and number of terms given, to find the sum of the series.

In this Case a, r, and n are given, and S required.

The last term can be found by Case I., and then the sum of the series by Case II. Or better, since

$$l = ar^{n-1}$$
$$lr = ar^n$$

Substituting this value of lr in the formula in Case II. we have

$$S = \frac{r^n - 1}{r - 1} \times a. \quad \text{Hence,}$$

RULE.

From the ratio raised to a power whose index is equal to the number of terms subtract one, divide the remainder by the ratio less one, and multiply the quotient by the first term.

1. Given $a = 4$, $r = 7$, and $n = 5$, to find S.

$$S = \frac{r^n - 1}{r - 1} \times a = \frac{7^5 - 1}{7 - 1} \times 4 = 11204, \text{ Ans.}$$

2. Given $a = \frac{1}{7}$, $r = 5$, and $n = 6$, to find S.
Ans. $S = 558$.

3. Given $a = \frac{1}{5}$, $r = \frac{1}{2}$, and $n = 7$, to find S.
Ans. $S = \frac{127}{320}$.

4. Given $a = -5$, $r = -4$, and $n = 4$, to find S.
Ans. $S = 255$.

5. Given $a = -\frac{1}{5}$, $r = 6$, and $n = 5$, to find S.

6. Given $a = \frac{2}{3}$, $r = -3$, and $n = 6$, to find S.

7. Given $a = -\frac{1}{7}$, $r = 2$, and $n = 8$, to find S.

236. In a geometrical series whose ratio is a proper fraction the greater the number of terms, the less, numerically, the last term. If the number of terms is infinite, the last term must be infinitesimal; and in finding the sum of such a series the last term may be considered as nothing. Therefore, when the number of terms is infinite, the formula

$$S = \frac{lr - a}{r - 1} \text{ becomes}$$

$$S = \frac{-a}{r - 1} = \frac{a}{1 - r}.$$

Hence, to find the sum of a geometrical series whose ratio is a proper fraction and number of terms infinite,

RULE.

Divide the first term by one minus the ratio.

1. Find the sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. to infinity.

$$S = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2, \text{ Ans.}$$

2. Find the sum of the series $\frac{3}{5}$, $\frac{1}{5}$, $\frac{1}{15}$, &c. to infinity.
Ans. $\frac{9}{10}$.

3. Find the sum of the series $\frac{1}{c}$, $\frac{1}{c^2}$, $\frac{1}{c^3}$, &c. to infinity.

Ans. $\frac{1}{c-1}$.

4. Find the sum of the series 6, 4, $2\frac{2}{3}$, &c. to infinity.
Ans. 18.

5. Find the value of the decimal .4444, &c. to infinity.

NOTE. — This decimal can be written $\frac{4}{10} + \frac{4}{100} + \frac{4}{1000}$, &c.
Ans. $\frac{4}{9}$.

6. Find the value of .324324, &c. to infinity.

7. Find the value of .32143214, &c. to infinity.

CASE IV.

237. The extremes and number of terms given, to find the ratio.

In this Case a, l, and n are given, and r is required.

From Case I. $\qquad l = ar^{n-1}$

Whence, $\qquad r = \sqrt[n-1]{\frac{l}{a}}$. Hence,

RULE.

Divide the last term by the first, and extract that root of the quotient whose index is one less than the number of terms.

1. Given $a = 7$, $l = 567$, and $n = 5$, to find r.

$$r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[4]{\frac{567}{7}} = 3, \text{ Ans.}$$

2. Given $a = 6\frac{2}{5}$, $l = \frac{1}{5}$, and $n = 6$, to find r.
Ans. $r = \frac{1}{2}$.

3. Given $a = -\frac{1}{4}$, $l = 31\frac{1}{4}$, and $n = 4$, to find r.
Ans. $r = -5$.

NOTE. — This rule enables us to insert any number of geometrical means between two numbers; for the number of terms is two greater than the number of means. Hence, if $m =$ the number of means, $m + 2 = n$, or $m + 1 = n - 1$; and $r = \sqrt[m+1]{\frac{l}{a}}$. Having found the ratio, the means are found by multiplying the first term by the *ratio*, by its *square*, its *cube*, &c.

4. Find three geometrical means between 2 and 512.

$$r = \sqrt[m+1]{\frac{l}{a}} = \sqrt[4]{\frac{512}{2}} = \sqrt[4]{256} = 4.$$

Ans. 8, 32, 128.

5. Find four geometrical means between 3 and 3072.
Ans. 12, 48, 192, 768.

6. Find three geometrical means between 1 and $\frac{1}{16}$.
Ans. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$.

NOTE. — When $m = 1$, the formula becomes

$$r = \sqrt{\frac{l}{a}}$$

Multiplying by a, $$a r = a\sqrt{\frac{l}{a}} = \sqrt{a l}.$$

But $a r$ is the second term of a series whose first term is a and ratio r; or the geometrical mean of the series a, $a r$, $a r^2$. Hence, *the geometrical mean between two quantities is the square root of their product.*

7. Find the geometrical mean between 8 and 18.
Ans. 12.

8. Find the geometrical mean between $\frac{1}{7}$ and 343.
Ans. 7.

9. Find the geometrical mean between $\frac{1}{5}$ and $\frac{1}{1125}$.

10. Find the geometrical mean between $-\frac{1}{3}$ and $-\frac{1}{2187}$.

238. From the formulas established in Arts. 233 and 234,

$$l = ar^{n-1} \tag{1}$$

$$S = \frac{lr - a}{r - 1} \tag{2}$$

can be derived formulas for all the Cases in Geometrical Progression.

From (1) we can obtain the value of any one of the four terms, l, a, n, or r, when the other three are given; from (2), the value of S, l, r, or a, when the other three are given. Formulas for the remaining twelve Cases which may arise are derived by combining the formulas (1) and (2) so as to eliminate that one of the two unknown terms whose value is not sought.

1. Find the formula for the value of S, when l, n, and r are given.

From (1),
$$\frac{l}{r^{n-1}} = a$$

Substituting this value of a in (2),
$$S = \frac{lr - \frac{l}{r^{n-1}}}{r - 1}$$

or
$$S = \frac{l\,(r^n - 1)}{(r - 1)\,r^{n-1}}$$

NOTE. — The four formulas for the value of n cannot be derived or used without a knowledge of logarithms; and four others, when n exceeds 2, cannot be reduced without a knowledge of equations that cannot be reduced by any rules given in this book.

239. To find any one of the five elements when three others are given.

RULE.

Substitute in that one of the formulas (1) *or* (2) *that contains the four elements, viz. the three given and the one required, the given values, and reduce the resulting equation.*

If neither formula contains the four elements, derive a formula that will contain them, then substitute and reduce the resulting equation; or substitute the given values before deriving the formula, then eliminate the superfluous element and reduce the resulting equation.

1. Given $r=3$, $n=5$, and $S=726$, to find l.

$$l=ar^{n-1} \qquad (1) \qquad\qquad S=\frac{lr-a}{r-1} \qquad (2)$$

$$l=81\,a \qquad (3) \qquad\qquad 726=\frac{3\,l-a}{2} \qquad (4)$$

$$\frac{l}{81}=a \qquad (5) \qquad\qquad a=3\,l-1452 \qquad (6)$$

$$3\,l-1452=\frac{l}{81} \qquad (7)$$

$$242\,l=1452\times 81 \qquad (8)$$

$$l=486 \qquad (9)$$

Substituting the given values of r, n, and S in (1) and (2), we obtain (3) and (4); finding the value of a, the superfluous element, from (3) and (4), and putting these values equal to each other, we form (7), an equation containing but one unknown quantity. Reducing (7) we obtain (9), or $l=486$.

2. Given $a=4$, $r=5$, and $S=15624$, to find l.
Ans. $l=12500$.

3. Given $a=2$, $n=5$, and $l=512$, to find S.
Ans. $S=682$.

4. Find the formula for the value of a, when r, n, and S are given.
Ans. $a=\frac{(r-1)\,S}{r^n-1}$.

5. A gentleman purchased a house, agreeing to pay one dollar if there was but one window, two dollars if there were two windows, four if there were three, and so on, doubling the price for every window. There were 14 windows. How much must he pay? Ans. \$8192.

6. A man found that a grain of wheat that he had sown had produced 10 grains. Now if he sows the 10 grains the next year, and continues each year to sow all that is produced, and it increases each year in tenfold ratio, how many grains will there be in the seventh harvest, and how many in all?

Ans. $\begin{cases}\text{In 7th harvest, } 10000000 \text{ grains.}\\ \text{In all, } 1111111 \text{ grains.}\end{cases}$

PROBLEMS

TO WHICH THE FORMULAS DO NOT DIRECTLY APPLY.

240. In solving Problems in Geometrical Progression, if we let $x =$ the first term and $y =$ the ratio, the series will be

$$x,\ xy,\ xy^2,\ xy^3,\ \&c.$$

It will often be found more convenient to represent the series in one of the following methods: —

1st. When the number of terms is odd,

$$\left.\begin{array}{c} x,\ \sqrt{xy},\ y \\ \text{or} \\ x^2,\ xy,\ y^2 \end{array}\right\} \text{ for three terms;}$$

$$\frac{x^3}{y},\ x^2,\ xy,\ y^2,\ \frac{y^3}{x} \text{ for five terms.}$$

2d. When the number of terms is even,

$$\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x} \text{ for four terms;}$$

$$\frac{x^3}{y^2},\ \frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x},\ \frac{y^3}{x^2} \text{ for six terms.}$$

Which method is most convenient in any case will depend upon the conditions that are given in the problem.

1. There are three numbers in geometrical progression, the greatest of which exceeds the least by 32; and the difference of the squares of the greatest and least is to the sum of the squares of the three as 80 : 91. What are the numbers?

SOLUTION.

Let x, xy, and xy^2 represent the series. Then

$$xy^2 - x = 32 \quad (1) \qquad x^2y^4 - x^2 : x^2 + x^2y^2 + x^2y^4 = 80 : 91 \quad (2)$$

$$y^4 - 1 : 1 + y^2 + y^4 = 80 : 91 \quad (3)$$

$$91\,y^4 - 91 = 80 + 80\,y^2 + 80\,y^4 \quad (4)$$

$$11\,y^4 - 80\,y^2 = 171 \quad (5)$$

$$x = 4 \quad (7) \qquad y = 3 \quad (6)$$

Dividing the first couplet of (2) by x^2, we obtain (3); from (3) we form (4), which reduced gives (6), or $y = 3$. Substituting the value of y in (1), we obtain (7), or $x = 4$.

Ans. 4, 12, 36.

2. The sum of three numbers in geometrical progression is 39, and the sum of their squares 819. What are the numbers?

SOLUTION.

Let x, $\sqrt{xy}$, and y represent the series. Then

$$x + \sqrt{xy} + y = 39 \quad (1) \qquad x^2 + xy + y^2 = 819 \quad (2)$$

$$x - \sqrt{xy} + y = 21 \quad (3)$$

$$2x + 2y = 60 \quad (4)$$

$$x + y = 30 \quad (5)$$

$$2\sqrt{xy} = 18 \quad (6)$$

$$xy = 81 \quad (7)$$

Dividing (2) by (1), we obtain (3); adding (3) to (1), we obtain (4), which reduced gives (5); subtracting (3) from (1), we obtain (6), which reduced gives (7). Combining (5) and (7) as the sum and product are combined in Example 1, Art. 188, we obtain $x = 27$ and $y = 3$.

Ans. 3, 9, 27.

3. Of four numbers in geometrical progression the difference between the fourth and second is 60; and the sum of the extremes is to the sum of the means as 13 : 4. What are the numbers?

SOLUTION.

Let x, xy, xy^2, and xy^3 represent the series. Then

$$xy^3 - xy = 60 \quad (1) \qquad xy^3 + x : xy^2 + xy = 13 : 4 \quad (2)$$

$$y^2 - y + 1 : y = 13 : 4 \quad (3)$$

$$4y^2 - 4y + 4 = 13y \quad (4)$$

$$64x - 4x = 60 \quad (7) \qquad 4y^2 - 17y = -4 \quad (5)$$

$$x = 1 \quad (8) \qquad y = 4 \quad (6)$$

Dividing the first couplet of (2) by $xy + x$, we obtain (3); from (3) we form (4), which reduced gives (6), or $y = 4$. Substituting this value of y in (1) and reducing, we obtain (8), or $x = 1$.

Ans. 1, 4, 16, 64.

4. Of four numbers in geometrical progression the sum of the first two is 10 and of the last two 160. What are the numbers? Ans. 2, 8, 32, 128.

5. A man paid a debt of \$310 at three payments. The several amounts paid formed a geometrical series, and the last payment exceeded the first by \$240. What were the several payments? Ans. \$10, \$50, \$250.

6. In the series x, $\sqrt{xy}$, and y what is the ratio?

Ans. $\sqrt{\frac{y}{x}}$.

7. In the series $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$ what is the ratio?

8. There are four numbers in geometrical progression whose continued product is 64; and the sum of the series is to the sum of the means as 5 : 2. What are the numbers? Ans. 1, 2, 4, 8.

9. There are five numbers in geometrical progression; the sum of the first four is 156, and the sum of the last four 780. What are the numbers?

10. There are three numbers in geometrical progression whose sum is 126; and the sum of the extremes is to the mean as 17 : 4. What are the numbers?

11. The sum of the squares of three numbers in geometrical progression is 2275; and the sum of the extremes is 35 more than the mean. What are the numbers?

12. Of four numbers in geometrical progression the sum of the first and third is 52; and the difference of the means is to the difference of the extremes as 5 : 31. What are the numbers?

SECTION XXIV.

MISCELLANEOUS EXAMPLES.

1. From $6ac - 5ab + c^2$ take $3ac - \{3ab - (c - c^2) + 7c\}$. Ans. $3ac - 2ab + 2c^2 + 6c$.

2. Reduce $x^3y^2 - (-xy^2 + x^3 - \frac{x^4}{y})\,xy - x^2\left(- \{y^3 - y(xy - x^2)\}\right)$ to its simplest form.

Ans. $2x^2y^3 + x^5$.

3. Reduce $(a - b + c)^2 - \left(a(c - a - b) - \{b(a + b + c) - c(a - b - c)\}\right)$ to its simplest form.

Ans. $2(a^2 + b^2 + c^2)$.

4. Reduce $(x + a)\,a + y - \{(y + b)(x + b) - y(x + a - 1) - (x + y)(b - a)\}$ to its simplest form.

Ans. $a^2 - b^2$.

5. Reduce $(a^2 - b^2)\,c - (a - b)\,\{a(b + c) - b(a - c)\}$ to its simplest form. Ans. 0.

6. Reduce $(a + b)\,x - (b - c)\,c - \{(b - x)\,b - (b - c)(b + c)\} - ax$ to its simplest form. Ans. $2bx - bc$.

7. Multiply $a^3 + 2a^2b - 3ab^2$ by $-(-3a^3b + a^2b^2)$.

8. Multiply $a^4 + 6a^2 + 9$ by $a^4 - 6a^2 + 9$.

9. Multiply $a + b - c$ by $a - b + c$.

10. Divide $28a^2 - 6a^3 - 6a^5 - 4a^4 - 96a + 264$ by $3a^2 - 4a + 11$.

11. Divide $1 - 18x^2 + 81x^4$ by $1 + 6x + 9x^2$.

12 Divide $9a^2 + 1 - 4a^4 - 6a$ by $1 + 2a^2 - 3a$.

13. Divide $9x^6 - 7x^2y^2 + 2y^3$ by $3x^4 + 2x^2y - y^2$.

14. Divide $23a - 30 - 7a^3 + 6a^4$ by $3a - 2a^2 - 5$.

15. Find the prime factors of $a^4 - b^4$.

16. Find the prime factors of $4m^6n^2 - 49m^4n^{10}$.

17. Find the prime factors of $x^2 - 2xy + y^2$.

18. Find the prime factors of $x^8 - y^8$.

19. Find the greatest common divisor of $5x^3 - 10x^2y + 15y^3$ and $4x^3 + 8x^2y + 8xy^2 + 4y^3$. Ans. $x + y$.

20. Find the greatest common divisor of $8ab^5 + 24ab^4 + 16ab$ and $7b^6 + 7b^5 + 7b^4 - 7b^2$. Ans. $b^2 + b$.

21. Find the greatest common divisor of $6x^2 + 7xy - 3y^2$ and $12x^2 + 22xy + 6y^2$.

22. Find the greatest common divisor of $4x + 4x^3 - 40$ and $3x^4y - 48y$. Ans. $x - 2$.

23. Reduce $\dfrac{(a^2 - b^2)(a + b)}{(a - b)(a^2 + 2ab + b^2)}$ to its lowest terms.

24. Reduce $\dfrac{a^3 - 3a^2b + 3ab^2 - b^3}{a^2 - b^2}$ to its lowest terms.

25. Reduce $\dfrac{x^4 - y^4}{(x^2 + y^2)(x^2 - 2xy + y^2)}$ to its lowest terms.

26. Find the least common denominator and reduce $\dfrac{1}{1 - a} - \dfrac{1}{1 + a} - \dfrac{a}{1 + a^2} - \dfrac{2a}{1 - a^2}$ to a single fraction.

Ans. $-\dfrac{a}{1 + a^2}$.

27. Find the least common denominator and reduce $\dfrac{1 + m^2}{1 - m^4} - \dfrac{1 - m^2}{1 + m^2}$ to a single fraction.

28. Find the least common denominator and reduce $\dfrac{4a^2 + 3ab}{4a^2 - 3ab} - 1 - \dfrac{48a^3b}{16a^4 - 9a^2b^2}$ to a single fraction.

29. Reduce to one fraction with the least possible denominator $\dfrac{a}{b} - \dfrac{b^2 - a^2 + ab}{bcd} - \dfrac{3b - a}{cd} + \dfrac{c}{bd}$.

30. Reduce to one fraction with the least possible denominator $\frac{a+b}{(b-c)(c-a)} - \frac{b+c}{(a-c)(a-b)} + \frac{a+c}{(b-a)(c-b)}$.

Ans. $\frac{0}{(b-c)(c-a)(a-b)} = 0$.

31. Find the least common denominator and reduce $\frac{3+2x}{2-x} - \left(\frac{2-3x}{2+x} - \frac{(16-x)x}{x^2-4}\right)$ to a single fraction.

Ans. $\frac{1}{2+x}$.

32. Reduce to one fraction with the least possible denominator $\frac{1+x}{(1-x)^2} - \frac{4x}{1-x^2} - \frac{1-x}{(1+x)^2}$. Ans. $\frac{2x+6x^3}{(1-x^2)^2}$.

33. Reduce $a - c - \frac{b-(c-d)e}{e}$ to its simplest form.

34. Reduce $\dfrac{1+\frac{n-1}{n+1}}{1-\frac{n-1}{n+1}}$ to its simplest form.

35. Reduce $\frac{2xy}{x-2y} + x$ and $\frac{2xy}{x+2y} - 2y$ each to a single fraction and find their product. Ans. $\frac{4x^2y^2}{4y^2-x^2}$.

36. Subtract $\frac{y}{y-x}$ from $\frac{x}{x+y}$.

37. Subtract $3x + \frac{x}{b}$ from $x - \frac{x-a}{c}$.

38. Multiply $\left(-\frac{2a}{b^4c^3}\right)^4$ by $\sqrt[3]{-\frac{b^{15}}{8a^{18}c^3}}$.

39. Divide $\frac{x^2}{a-x}$ by $\frac{a^2x^2}{a^2-x^2}$, and multiply the result by a^2.

40. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

41. Divide $\frac{a^2+b^2}{m+n}$ by $\left(\frac{a}{a+b} + \frac{b}{a-b}\right)$.

42. Divide $\frac{4(a^2 - ab)}{b(a+b)^2}$ by $\frac{6ab}{a^2 - b^2}$.

43. Divide $\frac{a+x}{a-x} + \frac{a-x}{a+x}$ by $\frac{a+x}{a-x} - \frac{a-x}{a+x}$, and give the answer in its lowest terms. Ans. $\frac{a^2+x^2}{2ax}$.

44. Reduce $\frac{x}{a} - \frac{a}{a+b} = \frac{x}{a-b}$. What is the value of x, if $a = -2$ and $b = 3$?

45. Reduce $x - \frac{x-1}{2} = \frac{x}{7} + \frac{3x-1}{2}$.

46. Reduce $(a+x)(b-x) - a(b-c) - \frac{a^2c - bx^2}{b} = 0$. What is the value of x, if $a = 2$, $b = -3$, and $c = -1$.

47. Reduce $\frac{a-2x}{b} = \frac{cx - bc}{a}$. What is the value of x, if $a = 2$, $b = -1$, and $c = 3$?

48. Reduce $a - \frac{1+x}{1-x} = 0$.

49. Find the value of x in the equation $x = \frac{\frac{ab}{a^2-b^2}}{\frac{a+b}{a-b} - \frac{a-b}{a+b}}$ in its simplest form.

50. A man spends \$2. He then borrows as much money as he has left, and again spends \$2. Then borrowing again as much money as he has left, he again spends \$2, and then has nothing left. How much money did he have at first?

51. If 5 is subtracted from a certain number, two thirds of the remainder will be 40. What is the number?

52. Having a certain sum of money in my pocket, I lost c dollars, and then spent one ath part of what remained and had left one bth part of what I had at first. What was the original sum? What does the answer become if $a = 3$, $b = 9$, and $c = 5$?

53. If I buy a certain number of pounds of beef at $0.25 a pound, I shall have $0.25 left; but if I buy the same number of pounds of lard at $0.15 a pound, I shall have $1.25 left. How much money have I?

54. Divide 84 into three parts so that one third of the first, one fourth of the second, and one fifth of the third shall be equal.

55. In a certain orchard 25 more than one fourth of the trees are apple trees, 2 less than one fifth are pear trees, and the rest, one sixth of the whole, are peach trees. How many trees are there in the orchard?

56. A merchant spent each year for three years one third of the stock which he had at the beginning of the year; during the first year he gained $600, the second $500, and the third $400. At the end of the three years he had but two thirds of his original stock. What was his original stock?

57. From a cask of wine out of which a third part had leaked, 84 liters were drawn, and then the cask was half full. What is the capacity of the cask?

58. A gentleman has two horses and a chaise. The chaise is worth a dollars more than the first horse and b dollars more than the second. Three fifths of the value of the first horse subtracted from the value of the chaise is the same as seven thirds of the value of the second horse subtracted from twice the value of the chaise. What is the value of the chaise and of each horse? What are the answers if $a = -50$ and $b = 50$?

59. A had twice as much money as B. A gained $30 and B lost $40. Then A gave B three tenths as much as B had left, and had left himself 20 per cent more than he had at first. How much did each have at first?

60. A number of men had done one third of a piece of work in 9 days, when 18 men were added and the work completed in 12 days. What was the original number of men?

61. A boatman can row down the middle of a river 14 miles in 2 hours and 20 minutes; but though he keeps near the shore where the current is one half as swift as in the middle, it takes him 4 hours and 40 minutes to row back. What is the velocity of the water in the middle of the river? Ans. 2 miles an hour.

62. A had three fifths as much money as B. A paid away \$80 more than one third of his, and B \$50 less than four ninths of his, when A had left one third as much as B. What sum had each at first?

63. A farmer hired a man and his son for 20 days, agreeing to pay the man \$3.50 a day and the son \$1.25 for every day the son worked; but if the son was idle, the farmer was to receive \$0.50 a day for the son's board. For the 20 days' labor the man received \$67. How many days did the son work?

64. I purchased a square piece of land and a lot of three-inch pickets to fence it. I found that if I placed the pickets 3 inches apart, I should have 50 pickets left; but if I placed the pickets $2\frac{1}{2}$ inches apart, I must purchase 50 more. How much land and how many pickets did I purchase? Ans. $18906\frac{1}{4}$ square feet and 1150 pickets.

65. A criminal having escaped from prison travelled 10 hours before his escape was discovered. He was then pursued and gained upon 3 miles an hour. When his pursuers had been on the way 8 hours, they met an expressman going at the same rate as themselves, who had met the criminal 2 hours and 24 minutes before. In what time from the commencement of the pursuit will the criminal be overtaken? Ans. 20 hours.

66. In February, 1868, a man being asked the time, answered that the number of hours before the close of the month was exactly one sixth of 10 less than the number that had passed in the month. What was the exact time?

Ans. February 25th, 10 o'clock, P. M.

67. A and B owned adjoining lots of land whose areas were as 3 : 4. A sold to B 100 hectares of his, and afterward purchased of B two fifths of B's entire lot; and then the original ratio of their quantities of land had been reversed. How much land did each own at first?

Ans. A, 300; B, 400 hectares.

68. A laborer was hired for 70 days; for each day he wrought he was to receive $2.25, and for each day he was idle he was to forfeit $0.75. At the end of the time he received $118.50. How many days did he work?

69. A sum of money was divided equally among a number of persons by giving to the first $100 and one sixth of the remainder, then to the second $200 and one sixth of the remainder, then to the third $300 and one sixth of the remainder; and so on. What was the sum divided and what the number of persons?

70. A besieged garrison had a quantity of bread which would last 9 days if each man received two hectograms a day. At the end of the first day 800 men were lost in a sally, and it was found that each man could receive $2\frac{2}{9}$ hectograms a day for the remainder of the time. What was the original number of men?

71. Find a fraction such that if 1 is added to the denominator its value will be $\frac{1}{2}$; but if the denominator is divided by 3 and the numerator diminished by 3, its value will be $\frac{2}{3}$.

72. If 7 years are added to A's age, he will be twice as old as B; but if 9 years are subtracted from B's age, he will be one third as old as A. What is the age of each?

73. A, B, and C compare their fortunes. A says to B, "Give me $700 of your money, and I shall have twice as much as you retain." B says to C, "Give me $1400, and I shall have three times as much as you retain." C says to A, "Give me $420, and I shall have five times as much as you retain." How much has each?

74. An artillery regiment had 39 soldiers to every 5 guns, and 4 over, and the whole number of soldiers and officers was six times the number of guns and officers. But after a battle in which the disabled were one half of those left fit for duty, there lacked 4 of being 22 men to every 4 guns. How many guns, how many officers, and how many soldiers were there?

Ans. 120 guns, 44 officers, 940 soldiers.

75. A car containing 5 more cows than oxen was started from Springfield to Boston. The freight for 4 oxen was $2 more than the freight for 5 cows, and the freight for the whole would have amounted to $30; but at the end of half the journey 2 more oxen and 3 more cows were taken into the car, in consequence of which the freight of the whole was increased in the proportion of 6 to 5. What was the original number of cows and oxen, and what was the freight for each?

Ans. { 9 cows and 4 oxen.
Freight for a cow, $2; for an ox, $3.

76. Two sums of money amounting together to $1600 were put at interest, the less sum at 2 per cent more than the other. If the interest of the greater sum had been increased 1 per cent, and the less diminished 1 per cent, the interest of the whole would have been increased one fifteenth; but if the interest of the greater had been increased 1 per cent while the interest of the other remained the same, the interest of the whole would have been increased one tenth. What were the sums, and the rates of interest?

Ans. $1200 at 7 per cent; $400 at 9 per cent.

77. A and B can perform a piece of work together in $17\frac{1}{7}$ days. They work together 10 days, and then B finishes the work alone in $16\frac{2}{3}$ days. How long would it take each to do the work?

78. The Emancipation Proclamation of President Lincoln was promulgated on the 1st day of January in a year represented by a number that has the following properties: the second (hundred's) figure is equal to the sum of the third and fourth minus the first; or to twice the sum of the first and fourth; the third is a third part of the sum of the four; and if 1818 is added to the number, the order of the figures will be inverted. What was the year?

79. A and B can do a piece of work in a days; A and C in b days; B and C in c days. In how many days can each do it?

80. A can do a piece of work in a days, B in b days, and C in c days. In how many days can A and B together do it? B and C together? A and C together? All three together?

81. A market-man bought some eggs for $0.28 a dozen, and sold some of them at 3 for 8 cents and some at 5 for 12 cents, receiving for the whole $6.24, and clearing $0.64. How many did he sell at each rate?

82. One cask contains 56 liters of wine and 40 of water, and another 96 of wine and 16 of water. How many liters taken from each cask will make a mixture containing 52 liters of wine and 24 of water?

83. A and B are travelling on roads which cross each other. When B is at the point of crossing, A has 720 meters to go before he arrives at this point, and in 4 minutes they are equally distant from this point; and in 32 minutes more they are again equally distant from it. What is the rate of each? Ans. A's, 100; B's, 80 meters a minute.

84. Multiply x^m by x^n.

85. Multiply x^7 by x^{-9}.

86. Divide y^{-4} by y^{-7}.

87. Divide a^{n-2} by a^{2+n}.

88. Transfer the denominator of $\frac{x^4 y^3}{a^{-1} b^3}$ to the numerator.

89. Free $\frac{a^{-3} b^{-4} c^3}{x^{-2} y^5 z^{-3}}$ from negative exponents.

90. Expand $(-2a^3)^4$.

91. Expand $(a^2 b)^m$.

92. Expand $(-3x^{-2}y^m)^4$.

93. Expand $(x^m \times x^n)^s$.

94. Expand $(x - \sqrt{y})^6$.

95. Expand $(a^2 - 2b)^5$.

96. Expand $(2x - y^2)^7$.

97. Expand $(2x - 3)^4$.

98. Expand $(3a - 2b)^8$.

99. Find five terms of $(x - y)^{25}$.

100. Expand $(2 - x - y)^3$.

101. Expand $(3 - a - b + c)^2$.

102. Find $\sqrt[3]{a^{18}}$.

103. Find $\sqrt{\frac{a^{m+n}}{a^{m-n}}}$.

104. Find $\sqrt{-16x^6}$.

105. Find the square root of $\frac{a}{b} - \left(2a + 2x - 4\right)\sqrt{\frac{a}{b}}$ $+ a^2 - 4a + 2ax + 4 - 4x + x^2$.

106. Reduce $\sqrt[4]{256a^4b^8 - 768a^5b^9c^7}$ to its simplest form

107. Reduce $\sqrt[3]{\frac{2x}{3}}$ and $\sqrt[5]{\frac{y}{2}}$ to equivalent radicals having a common index.

108. Add $\sqrt{\frac{3}{16}}$, $\sqrt{75}$, and $\sqrt{\frac{12}{49}}$.

109. From $\sqrt[3]{135}$ take $\sqrt[3]{40}$.

110. Multiply $\frac{1}{6}\sqrt[3]{\frac{2}{3}}$ by $\frac{1}{5}\sqrt[3]{\frac{4}{9}}$.

111. Divide $-3\sqrt{10}$ by $\sqrt{5}$.

112. Divide $\sqrt[3]{6}$ by $\sqrt[4]{6}$.

113. Find the cube of $3\sqrt{2x}$.

114. Find the square root of $6\sqrt[3]{3}$.

115. Multiply $6+\sqrt{3}$ by $3-\sqrt{3}$.

116. Expand $(x^2-2\sqrt{x})^5$.

117. Expand $\left(\sqrt{\frac{x}{y}}-y^{-1}\right)^4$.

118. Expand $\left(\sqrt{\frac{a}{b}}-\frac{a}{3}\right)^5$.

119. The area of a rectangular field is 4 acres and 35 square rods; and the sum of its length and breadth is equal to twice their difference. What are the length and breadth?

120. Two travellers, A and B, set out to meet each other. They started at the same time and travelled on the direct road toward each other. On meeting it appeared that A had travelled 18 miles more than B, and that A could have travelled B's distance in 9 days, while it would have taken B 16 days to travel A's distance. How far did each travel? Ans. A, 72 miles; B, 54 miles.

121. Find three quantities such that the product of the first and second is a; of the second and third, b; and of the first and third, c.

Ans. $\pm\sqrt{\frac{ac}{b}}$, $\pm\sqrt{\frac{ab}{c}}$, $\pm\sqrt{\frac{bc}{a}}$.

122. A and B invest in stocks. At the end of the year A sells his stocks for \$108, gaining as much per cent as B invested; B sold his for \$49 more than he paid, gaining one fourth as much per cent as A. What sum did each invest? Ans. A, \$45; B, \$140.

123. Reduce $18x^2 - 33x - 40 = 0$.

124. Reduce $\frac{x}{x-1} - \frac{x+1}{x} = \frac{1}{6}$.

125. Reduce $\left(\frac{9}{y} - y\right)^2 - \left(\frac{9}{y} - y\right) = \frac{336}{25}$.

Ans. $y = -2.1 \pm 3\sqrt{1.49}$, or 5, or $-\frac{9}{5}$.

126. Reduce $(x^4 - x^2 + 4)^2 + x^2 = 5704 + x^4$.

Ans. $x = \pm 3$, or $\pm 2\sqrt{-2}$, or $\pm\sqrt{\frac{1 \pm 3\sqrt{-35}}{2}}$.

127. Reduce $\sqrt{2x+1} + 2\sqrt{x} = \frac{21}{\sqrt{2x+1}}$.

128. Reduce $y^2 - 2\sqrt{y^2 - 3y + 5} = 3y - 2$.

129. Reduce $\frac{2\sqrt{x} - 5}{\sqrt{x} + 2} = \frac{\sqrt{x} - 2}{\sqrt{x}}$.

130. Reduce $\frac{x^3 - 5x^2 + 24}{x^2 - 4x + 4} = x - 2$.

131. Reduce $\left(\frac{x + \sqrt{x^2 - 16}}{x - \sqrt{x^2 - 16}}\right)^{\frac{1}{2}} = x - 3$.

132. Given $\left\{\begin{array}{r} 3xy = 18 \\ x^2 + y^2 = 13 \end{array}\right\}$, to find x and y.

133. Given $\left\{\begin{array}{r} x + y = 10 \\ x^2 + y^2 = 52 \end{array}\right\}$, to find x and y.

134. Given $\left\{\begin{array}{r} \frac{x^3 - y^3}{x - y} = 49 \\ 5xy = 75 \end{array}\right\}$, to find x and y.

135. Given $\begin{cases} x^3 - y^3 = 218 \\ 2x^2y - 2xy^2 = 140 \end{cases}$, to find x and y.

136. Given $\begin{cases} 5x^2y - 2xy^2 = 875 \\ 3xy = 105 \end{cases}$, to find x and y.

137. Given $\begin{cases} x^2y^2 - 4xy = 96 \\ x^2 + y^2 = 25 \end{cases}$, to find x and y.

138. Given $\begin{cases} 5x^2 - 3xy = 182 \\ 3y^2 + 5xy = 132 \end{cases}$, to find x and y.

139. Given $\begin{cases} x^2 + y^2 + x + y = 12 \\ x + y - xy = 0 \end{cases}$, to find x and y.

Ans. $\begin{cases} x = 2, \text{ or } \frac{1}{2}(-3 \pm \sqrt{21}). \\ y = 2, \text{ or } \frac{1}{2}(-3 \mp \sqrt{21}). \end{cases}$

140. Given $\begin{cases} x + y = 11 \\ x^4 + y^4 = 1921 \end{cases}$, to find x and y.

141. A drover sold a number of sheep that cost him \$297 for \$7 each, gaining \$3 more than 36 sheep cost him. How many sheep did he sell?

142. A merchant sold a piece of cloth for \$75, gaining as much per cent as the piece cost him. What did it cost him?

143. A drover bought 12 oxen and 20 cows for \$920, buying one ox more for \$160 than cows for \$66. What did he pay a head for each?

144. A started from C towards D and travelled 4 miles an hour. After A had been on the road $6\frac{1}{4}$ hours, B started from D towards C, and travelled every hour one fourteenth of the whole distance, and after he had been on the road as many hours as he travelled miles an hour, he met A. What was the distance from C to D?

145. A person bought a number of horses for $1404. If there had been 3 less, each would have cost him $39 more. What was the number of horses and the cost of each?

146. Find a number of four figures which increase from left to right by a common difference 2, while the product of these figures is 384. Ans. 2468.

147. A rectangular garden 24 rods in length and 16 in breadth is surrounded by a walk of uniform breadth which contains 3996 square feet. What is the breadth of the walk? Ans. 3 feet.

148. A square field containing 144 ares has just within its borders a ditch of uniform breadth running entirely round the field and covering 381.44 centares of the area. What is the breadth of the ditch? Ans. 0.8 meter.

149. A and B hired a pasture into which A put 5 horses, and B as many as cost him $5.50 a week. If B had put in 4 more horses, he ought to have paid $6 a week. What was the price of the pasture a week? Ans. $8.

150. A father dying left $3294 to be divided equally among his children. Had there been 3 children less, each would have received $183 more. How many children were there?

151. A merchant bought a quantity of tea for $66. If he had invested the same sum in coffee at a price $0.77 less a pound, he would have received 140 pounds more. How many pounds of tea did he buy?

152. Find two quantities such that their sum, product, and the sum of their squares shall be equal to one another. Ans. $\frac{1}{2}(3 \pm \sqrt{-3})$ and $\frac{1}{2}(3 \mp \sqrt{-3})$.

153. Find two numbers such that their product shall be 6, and the sum of their squares 13.

154. A and B talking of their ages find that the square of A's age plus twice the product of the ages of both is 3864; and four times this product, minus the square of B's age, is 3575. What is the age of each?

Ans. A's, 42; B's, 25.

155. Find two numbers such that five times the square of the less minus the square of the greater shall be 20; and five times their product minus twice the square of the greater shall be 25.

156. A and B purchased a wood-lot containing 600 acres, each agreeing to pay $17500. Before paying for the lot, A offered to pay $20 an acre more than B, if B would consent to a division and give A his choice of situation. How many acres should each receive, and at what price an acre?

Ans. A, 250 acres at $70 an acre; B, 350 at $50.

157. A merchant bought two pieces of cloth for $175. For the first piece he paid as many dollars a yard as there were yards in both pieces; for the second, as many dollars a yard as there were yards in the first more than in the second; and the first piece cost six times as much as the second. What was the number of yards in each piece? Ans. In 1st, 10 yards; in 2d, 5.

158. Two sums of money amounting to $14300 were lent at such a rate of interest that the income from each was the same. But if the first part had been at the same rate as the second, the income from it would have been $532.90; and if the second part had been at the same rate as the first, the income from it would have been $490. What was the rate of interest of each?

Ans. First, 7 per cent; second, $7\frac{3}{10}$ per cent.

159. Divide 29 into two such parts that their product will be to the sum of their squares as 198 : 445.

160. What is the length and breadth of a rectangular field whose perimeter is 10 rods greater than a square field whose side is 50 rods, while its area is 250 square rods less than the area of the square field?

Ans. Length, 75 rods; breadth, 30.

161. A rectangular piece of land was sold for $5 for every rod in its perimeter. If the same area had been in the form of a square, and sold in the same way, it would have brought $90 less; and a square field of the same perimeter would have contained $272\frac{1}{4}$ square rods more. What were the length and breadth of the field?

Ans. Length, 49; breadth, 16 rods.

162. A starts from Springfield to Boston at the same time that B starts from Boston to Springfield. When they met, A had travelled 30 miles more than B, having gone as far in $1\frac{3}{4}$ days as B had during the whole time; and at the same rate as before B would reach Springfield in $5\frac{1}{7}$ days. How far from Boston did they meet?

Ans. 42 miles.

163. The product of two numbers is 90; and the difference of their cubes is to the cube of their difference as 13 : 3. What are the numbers?

164. A and B start together from the same place and travel in the same direction. A travels the first day 25 kilometers, the second 22, and so on, travelling each day 3 kilometers less than on the preceding day, while B travels $14\frac{1}{2}$ kilometers each day. In what time will the two be together again? Ans. 8 days.

165. A starts from a certain point and travels 5 miles the first day, 7 the second, and so on, travelling each day 2 miles more than on the preceding day. B starts from the same point 3 days later and follows A at the rate of 20 miles a day. If they keep on in the same line, when will they be together? Ans. 3 or 7 days after B starts.

166. A gentleman offered his daughter on the day of her marriage \$1000; or \$1 on that day, \$2 on the next, \$3 on the next, and so on, for 60 days. The lady chose the first offer. How much did she gain, or lose, by her choice?

167. The arithmetical mean of two numbers exceeds the geometrical mean by 2; and their product divided by their sum is $3\frac{1}{5}$. What are the numbers?

168. A father divided \$130 among his four children in arithmetical progression. If he had given the eldest \$25 more and the youngest but one \$5 less, their shares would have been in geometrical progression. What was the share of each?

169. The sum of the squares plus the product of two numbers is 133; and twice the arithmetical mean plus the geometrical mean is 19. What are the numbers?

170. The sum of three numbers in geometrical progression is 117; and the difference of the second and third minus the difference of the first and second is 36. What are the numbers?

171. There are four numbers in geometrical progression, and the sum of the second and fourth is 60; and the sum of the extremes is to the sum of the means as 7 : 3. What are the numbers?

SECTION XXV.

LOGARITHMS.

241. LOGARITHMS are exponents of the powers of some number which is taken as a *base.* In the tables of logarithms in common use the number 10 is taken as the base, and all numbers are considered as powers of 10.

By Arts. 119, 120,

$10^0=1$,	that is, the logarithm of	1	is	0
$10^1=10$,	" "	10	"	1
$10^2=100$,	" "	100	"	2
$10^3=1000$,	" "	1000	"	3
&c.,	&c.,	&c.		

Therefore, the logarithm of any number between 1 and 10 is between 0 and 1, that is, is a fraction; the logarithm of any number between 10 and 100 is between 1 and 2, that is, is 1 plus a fraction; and the logarithm of any number between 100 and 1000 is 2 plus a fraction; and so on.

By Art. 120,

$10^0 = 1$,	that is, the logarithm of	1.	is	0
$10^{-1}=0.1$,	" "	0.1	"	-1
$10^{-2}=0.01$,	" "	0.01	"	-2
$10^{-3}=0.001$,	" "	0.001	"	-3
&c.,	&c.,	&c.		

Therefore, the logarithm of any number between 1 and 0.1 is between 0 and -1, that is, is -1 plus a fraction; the logarithm of any number between 0.1 and 0.01 is between -1 and -2, that is, is -2 plus a fraction; and so on.

The logarithm of a number, therefore, is either an integer (which may be 0) positive or negative, or an integer positive or negative and a fraction, which is always positive.

The representation of the logarithms of all numbers less than a unit by a *negative integer* and a *positive fraction* is merely a matter of convenience. The integral part of a logarithm is called the *characteristic*, and the decimal part the *mantissa*. Thus, the characteristic of the logarithm 3.1784 is 3, and the mantissa .1784.

242. The characteristic of the logarithm of a number is not given in the tables, but can be supplied by the following

RULE.

The characteristic of the logarithm of any number is equal to the number of places by which its first significant figure on the left is removed from units' place, the characteristic being positive when this figure is to the left and negative when it is to the right of units' place.

Thus, the logarithm of 59 is 1 plus a fraction; that is, the characteristic of the logarithm of 59 is 1. The logarithm of 5417.7 is 3 plus a fraction; that is, the characteristic of the logarithm of 5417.7 is 3. The logarithm of 0.3 is —1 plus a fraction; that is, the characteristic of the logarithm of 0.3 is —1. The logarithm of 0.00017 is —4 plus a fraction; that is, the characteristic of the logarithm of 0.00017 is —4.

243. Since the base of this system of logarithms is 10, if any number is multiplied by 10, its logarithm will be increased by a unit (Art. 50); if divided by 10, diminished by a unit (Art. 54).

That is, the log	of	5549	being	3.7442
"	"	554.9	is	2.7442
"	"	55.49	"	1.7442
"	"	5.549	"	0.7442
"	"	.5549	"	$\bar{1}.7442$
"	"	.05549	"	$\bar{2}.7442$
"	"	.005549	"	$\bar{3}.7442$

Hence, the mantissa of the logarithm of any set of figures is the same, wherever the decimal point may be.

As only the characteristic is negative, the minus sign is written *over* the characteristic.

TABLE OF LOGARITHMS.

244. To find the logarithm of a number of two figures.

Disregarding the decimal point, find the given number in the column **N** (pp. 268, 269), and directly opposite, in the column **O**, is the mantissa of the logarithm, to which must be prefixed the characteristic, according to the Rule in Art. 242.

Thus, the log of 85 is 1.9294
" " 26 " 1.4150

The first figure of the mantissa, remaining the same for several successive numbers, is not repeated, but left to be supplied.

Thus, the log of 83 is 1.9191

As, according to Art. 243, multiplying a number by 10 increases its logarithm by a unit, therefore, to find the logarithm of any number containing only two significant figures with one or more ciphers annexed, we use the same rule as above.

Thus, the log of 850 is 2.9294
" " 750000 " 5.8751

The principle just stated is applicable also in the cases that follow.

245. To find the logarithm of a number of three figures.

Disregarding the decimal point, find the first two figures in the column **N**, and the third figure at the top of one of the columns. Opposite the first two figures, and in the column under the third figure, will be the last three figures of the decimal part of the logarithm, to which the first figure in the

column **0** is to be prefixed, and the characteristic, according to the Rule in Art. 242.

Thus, the log of 295 is 2.4698
" " 549 " 2.7396

In the columns **1**, **2**, **3**, &c., a small cipher ($_0$) or figure ($_1$) is sometimes placed below the first figure, to show that the figure which is to be prefixed from the column **0** has changed to the next larger number, and is to be found in the horizontal line directly below.

Thus, the log of 7960 is 3.9009
" " 25900 " 4.4133

246. To find the logarithm of a number of more than three figures.

On the right half of pages 268 and 269 are tables of Proportional Parts. The figures in any column of these tables are as many tenths of the average difference of the ten logarithms in the same horizontal line as is denoted by the number at the top of the column. The decimal point in these differences is placed as though the mantissas were integral.

1. To find the logarithm of a number of four figures, find as before the logarithm of the first three figures; to this, from the table of Proportional Parts, add the number standing on the same horizontal line and directly under the fourth figure of the given number.

Thus, to find the log of 5743.

The log of 5740 is 3.7589
In "Proportional Parts," in the same line, under 3, " 2.3
Therefore, the log of 5743 " 3.7591

It is always best to find the logarithm of the nearest tabulated number, and add or subtract, as the case may be, the correction from the table of Proportional Parts.

Thus, to find the log of 6377.

$$6377 = 6380 - 3$$

The log of	6380	is	3.8048
correction for	3	"	2
Therefore, the log of	6377	"	3.8046

Whenever the fractional part omitted is larger than half the unit in the next place to the left, one is added to that figure.

2. For a fifth or sixth figure the correction is made in the same manner, only the point must be moved one place to the left for the fifth, two for the sixth, figure.

Thus, to find the log of 3.6825.

The log of	3.68	is	0.5658
correction for	2	"	2.4
" "	5	"	.59
Therefore, the log of	3.6825	"	0.5661

To find the log of 112.82.

$$112.82 = 113 - 0.18$$

The log of	113	is	2.0531
correction for .18 is	(3.8 + 3.02)	"	6.82
Therefore, the log of	112.82	"	2.0524

The logarithm of a common fraction may best be found by reducing the fraction to a decimal, and then proceeding as above.

247. To find the number corresponding to a given logarithm.

Find, if possible, in the table the mantissa of the given logarithm. The three figures opposite in the column **N**, with the number at the head of the column in which the logarithm is found, affixed, and the decimal point so placed as to make the number of integral figures correspond to the

characteristic of the given logarithm, as taught in Art. 242, will be the number required. Thus,

The number corresponding to log 5.5378 is 345000
" " " 1.8745 " 74.9

If the mantissa of the logarithm cannot be exactly found, take the number corresponding to the mantissa nearest the given mantissa; in the same horizontal line in the table of proportional parts find the figures which express the difference between this and the given mantissa; at the top of the page, in the same vertical column, is the correction that belongs one place to the right of the number already taken, — to be added if the given mantissa is greater, subtracted if less. Thus,

1. To find the number corresponding to

log	2.7660		
next less log,	2.7657,	and number corresponding,	583.
difference,	3	correction,	0.4
Number required,			583.4

2. To find the number corresponding to

log	$\bar{3}.8052$		
next greater log,	$\bar{3}.8055$,	and number corresponding,	0.00639
difference,	3	correction,	44
Number required,			0.0063856

The nearest number in the table of Proportional Parts to 3 is 2.7; corresponding to this at the top is 4, which belongs as a correction one place to the right of the number (0.00639) already taken, but $3 - 2.7 = 0.3$; this, in like manner, gives a still further correction of 4, one place farther still to the right. The whole correction, therefore, is 44, to be deducted as shown in the operation above.

3. Find the log of 3764.
4. Find the log of 2576000.
5. Find the log of 7.546.
6. Find the log of 0.0017.

7. Find the log of $\frac{4}{17}$.
8. Find the number to log 3.807873.
9. Find the number to log 1.820004.
10. Find the number to log 2.982197.
11. Find the number to log $\bar{2}.910037$.
12. Find the number to log $\bar{4}.850054$.

248. The great utility of logarithms in arithmetical operations is that addition takes the place of multiplication, and subtraction of division, multiplication of involution, and division of evolution. That is, to multiply numbers, we add their logarithms; to divide, we subtract the logarithm of the divisor from that of the dividend; to raise a number to any power, we multiply its logarithm by the exponent of that power; and to extract the root of any number, we divide its logarithm by the number expressing the root to be found.

This is the same as multiplication and division of different powers of the same letter by each other, and involving and evolving powers of a single letter or quantity; the number 10 takes the place of the given letter, and the logarithms are the exponents of 10.

MULTIPLICATION BY LOGARITHMS.

RULE.

249. *Add the logarithms of the factors, and the sum will be the logarithm of the product* (Art. 50).

1. Multiply 347.6 by 0.04752. Ans. 16.517.

2. Find the product of 0.568, 0.7496, 0.0846, and 1.728. Ans. 0.06224.

(It must be carefully borne in mind that the mantissa of the logarithm is *always* positive.)

3. Multiply 0.00756 by 17.5.

	0.00756	log $\bar{3}.8785$
	17.5	" 1.2430
Product,	0.1323	" $\bar{1}.1215$

4. Multiply 0.0004756 by 1355.

Although negative quantities have no logarithms (Art. 262), yet, since the *numerical* product is the same whether the factors are positive or negative, we can use logarithms in multiplying when one or more of the factors are negative, taking care to prefix to the product the proper sign according to Art. 48. When a factor is negative, to the logarithm which is used *n* is appended.

5. Multiply —0.7546 by 0.00545.

	—0.7546	log $\bar{1}.8777$ n
	0.00545	" $\bar{3}.7364$
Product,	—0.004113	" $\bar{3}.6141$ n

6. Find the product of —0.017, 25, and —165.4.

7. Find the product of —14, —7.643, and —0.004.

Ans. —.428.

DIVISION BY LOGARITHMS.

RULE.

250. *From the logarithm of the dividend subtract the logarithm of the divisor, and the remainder will be the logarithm of the quotient* (Art. 54).

1. Divide 78.46 by 0.00147.

	78.46	log 1.8946
	0.00147	" $\bar{3}.1673$
Quotient,	53374.	" 4.7273

2. Divide 0.0014 by 756.

0.0014	log	$\bar{3}.1461$
756	"	2.8785
Quotient, 0.000001852.	"	$\bar{6}.2676$

Negative numbers can be divided in the same manner as positive, taking care to prefix to the quotient the proper sign, according to Art. 53.

3. Divide 0.7478 by 0.00456. Ans. 164.

4. Divide 5000 by 0.00149.

5. Divide 0.00997 by 64.16. Ans. 0.0001554.

6. Divide —14.55 by 543. Ans. —0.0268.

7. Divide —465 by —19.45. Ans. 23.9.

251. Instead of subtracting one logarithm from another, it is sometimes more convenient to add what it lacks of 10, and from the sum reject 10. The result is evidently the same. For

$$x-y=x+(10-y)-10$$

The remainder found by subtracting a logarithm from 10 is called the *arithmetical complement* of the logarithm, or the *cologarithm.* The cologarithm is easiest found by beginning at the left of the logarithm, and subtracting each figure from 9, except the last significant figure, which must be subtracted from 10.

By this method, Ex. 1 will appear as follows:

78.46	log	1.8946
0.00147	colog	12.8327
Quotient, 53374.	log	4.7273

DIVISION AND MULTIPLICATION BY LOGARITHMS.

252. In working examples combining multiplication and division, the use of cologarithms is of great advantage.

RULE.

Find the sum of the logarithms of the multipliers and the cologarithms of the divisors; reject as many tens as there are cologarithms (divisors); the result will be the logarithm of the number sought.

1. Find the value of $\frac{673 \times 0.319 \times (-0.04)}{(-7.95) \times (-0.03478)}$

673.	log	2.8280
0.319	"	$\bar{1}.5038$
—0.04	"	$\bar{2}.6021$ n
—7.95	colog	9.0996 n
—0.03478	"	11.4587 n
Ans. —31.06		1.4922 n

An *even* number of negative quantities gives a positive result, an *odd* number a negative (Art. 48).

2. Find the value of $\frac{(-0.259) \times 8.7 \div 0.13}{-7.64 \times 0.657}$ Ans. 3.453.

3. Find the value of $\frac{8.17 \times 0.468}{7.99 \times (-0.631) \times (-1)}$ Ans. 0.758.

PROPORTION BY LOGARITHMS.

RULE.

253. *Add the cologarithm of the first term to the logarithms of the second and third terms, and from the sum reject* 10.

1. Given $14 : 175 = 7486 : x$, to find x.

14	colog	8.8539
175	log	2.2430
7486	"	3.87425
Ans. 93575	"	4.97115

2. Given $416 : 584 = 256 : x$, to find x. Ans. 359.3+

3. Given $x : 179 = 49.68 : 489$, to find x. Ans. 18.18+

INVOLUTION BY LOGARITHMS.

RULE.

254. *Multiply the logarithm of the number by the exponent of the power required* (Art. 126).

In involution, as the error in the logarithm is multiplied by the index of the power, the results with logarithms of only four decimal places cannot be relied on for more than two or three significant figures.

1. Find the 15th power of 1.17.

1.17	log	0.0682
		15
Ans. 10.54	"	1.0230

2. Find the 5th power of 0.00941.

0.00941	log	$\bar{3}.9736$
		5
Ans. 0.0000000000738	"	$\overline{11}.8680$

3. Find the 4th power of 0.0176. Ans. 0.00000009595.

4. Find the 9th power of 1.179. Ans. 4.49.

Negative numbers are involved in the same manner, taking care to prefix to the power the proper sign, according to Art. 125.

5. Find the 3d power of —0.017. Ans. —0.000004913.

6. Find the 6th power of —14. Ans. 7529536.

In the last two examples the *exact* answers are given, though from the table only answers approximating to these can be obtained.

EVOLUTION BY LOGARITHMS.

RULE.

255. *Divide the logarithm of the number by the exponent of the root required* (Art. 143).

When the characteristic is negative, and not divisible by the index of the root, we increase the negative characteristic so as to make it divisible, and to the mantissa prefix an equal positive number.

1. Find the 5th root of 0.0173.

$$0.0173 \quad \log \quad \bar{2}.2380 = 5)\,\bar{5} + 3.2380$$

Ans. 0.4442 " $\bar{1}.6476$

2. Find the 3d root of 80.07. Ans. 4.31.

3. Find the 8th root of 0.0764. Ans. ± 0.725.

Negative numbers are evolved in the same manner, taking care to prefix to the root the proper sign, according to Art. 136.

4. Find the 7th root of —17. Ans. —1.499.

5. Find the 5th root of —0.00496. Ans. —0.346.

MISCELLANEOUS EXAMPLES.

256. Verify the following expressions:

1. $\dfrac{1748 \times 917}{654 \times 513} = 4.777+$

2. $\dfrac{-0.03479 \times 2.3468^3 \times \sqrt[7]{6.843}}{\sqrt{0.0678^5} \times (-4.63) \times 78.56} = 1.359+$

3. $\sqrt[5]{\dfrac{-0.074\dot{2}8 \times \sqrt[3]{8.984}}{\sqrt[11]{673.21} \times 50.3 \times \sqrt[9]{0.6845}}} = -0.2816+$

4. $\sqrt[99]{\left(\dfrac{0.3476^4 \times 0.07996^3}{\sqrt[5]{0.00017} \times \sqrt[7]{0.00004782}}\right)^{103}} = 0.0001236+$

5. $\dfrac{\sqrt{0.1739}}{331.9\,(\sqrt{2.04} + \sqrt{1.203})^2} = 0.000197+$

6. $\dfrac{23.3 \times 6.764 \times \dfrac{85.31}{253.4}}{\sqrt[3]{47.64}\left(\dfrac{2.768}{9.853}\right)^5 \times 9.97} = 838.8+$

SYSTEMS OF LOGARITHMS.

257. The system of logarithms which has 10 for its base is the one in common use. As in this system the mantissa of the logarithm of any set of figures is the same, wherever the decimal point may be (Art. 243), which (in the Arabic notation of numbers) would not be the case with any other base, it is far the most convenient system. The number of possible systems, however, is infinite.

In general, if $a^x = n$, then x is the logarithm of n to the base a; and n is the number (sometimes called the *antilogarithm*) corresponding to the logarithm x, in a system whose base is a.

258. *The logarithm of* 1 *is* 0, *whatever the base may be.*

For the 0 power of every quantity is 1, or $a^0 = 1$ (Art. 120).

259. *The logarithm of the base itself is* 1.

For the first power of any quantity is that quantity itself, or $a^1 = 1$ (Art. 119).

260. *Neither* 0 *nor* 1 *can be the base of a system of logarithms.*

For all the powers and roots of 0 are 0, and of 1 are 1.

261. *The logarithm of the reciprocal of any quantity is the negative of the logarithm of the quantity itself.*

For the reciprocal of any quantity is 1 divided by that quantity (Art. 27); that is, is the logarithm of 1 minus the logarithm of the quantity; or 0 minus the logarithm of the quantity (Art. 250).

262. *In a system whose base is greater than* 1, *the logarithm of infinity* (∞) *is infinity; and the logarithm of* 0 *is minus infinity* ($-\infty$).

For $a^\infty = \infty$; and $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$.

Hence, negative quantities cannot have logarithms.

263. *In a system whose base is between* 1 *and* 0, *the less the number the greater its logarithm.*

For the greater the power of a proper fraction the less its

value. With such a base the logarithms of numbers greater than 1 will be negative, less than 1 positive,

Thus, with $\frac{1}{3}$ as the base,

the log of $\frac{1}{9}$ is 2; of $\frac{1}{81}$ is 3
" 9 " -2; " 81 " -3

264. *The logarithms of numbers which form a geometrical series form an arithmetical series.*

For, if a series increased or decreased by a constant ratio, its logarithms would increase or decrease by a constant difference equal to the logarithm of the constant ratio.

For an example see Art. 243; here the numbers decrease by the constant ratio 10, and the logarithms by the constant difference 1.

265. From the principles of the previous articles it will be easy to find the logarithms of the perfect roots and powers of any number. Thus,

1. In a system whose base is 8,

$8^{\frac{1}{3}} = 2$, that is, the log of $2 = 0.\dot{3}$
$8^{\frac{2}{3}} = 4$, " " $4 = 0.\dot{6}$
$8^{1} = 8$, " " $8 = 1.$
$8^{\frac{4}{3}} = 16$, " " $16 = 1.\dot{3}$
$8^{\frac{5}{3}} = 32$, " " $32 = 1.\dot{6}$
$8^{2} = 64$, " " $64 = 2.$
$8^{\frac{7}{3}} = 128$, " " $128 = 2.\dot{3}$
&c., &c., &c.

Then, according to Art. 261,

the log of $\frac{1}{2} = -0.\dot{3} = \bar{1}.\dot{6}$
" $\frac{1}{4} = -0.\dot{6} = \bar{1}.\dot{3}$
" $\frac{1}{8} = -1. = \bar{1}.$
" $\frac{1}{16} = -1.\dot{3} = \bar{2}.\dot{6}$
" $\frac{1}{32} = -1.\dot{6} = \bar{2}.\dot{3}$
" $\frac{1}{64} = -2. = \bar{2}.$
" $\frac{1}{128} = -2.\dot{3} = \bar{3}.\dot{6}$
&c.

2. In a system whose base is 4, what is the logarithm of 4? of 16? of 64? of 2? of 8? of 1? of $\frac{1}{2}$? of $\frac{1}{4}$? of $\frac{1}{8}$? of 0?

3. In a system whose base is 9, what is the logarithm of 81? of 3? of 27? of 9? of 1? of $\frac{1}{9}$? of $\frac{1}{81}$? of 0?

4. In a system whose base is $\frac{1}{2}$, what is the logarithm of 2? of 32? of 8? of $\frac{1}{4}$? of $\frac{1}{16}$?

5. If the logarithm of 0.125 is $\bar{2}.5$, what is the base?

$$x^{-\frac{3}{2}} = 0.125$$

$$x = 0.125^{-\frac{2}{3}} = \left(\frac{1}{8}\right)^{-\frac{2}{3}} = 8^{\frac{2}{3}} = 4, \text{ Ans.}$$

6. If the logarithm of 0.5 is $\bar{1}.8$, what is the base?

Ans. 32.

7. If the logarithm of $0.\dot{3}$ is $0.\dot{3}$, what is the base?

EXPONENTIAL EQUATIONS.

266. An equation having the unknown quantity as an exponent, or an *exponential equation*, may be solved by means of logarithms.

For, if $a^x = n$, by Art. 254,

$$x \times \log a = \log n$$

$$x = \frac{\log n}{\log a}$$

1. Solve the equation $125^x = 25$.

$$x \times \log 125 = \log 25$$

$$x = \frac{\log 25}{\log 125} = \frac{1.3979}{2.0969} = \frac{2}{3}, \text{ Ans.}$$

2. Solve the equation $2048^x = 16$.

3. Solve the equation $\left(\frac{1}{2187}\right)^x = 27$.

N	0	1	2	3	4	5	6	7	8	9	PROPORTIONAL PARTS. 1	2	3	4	5	6	7	8	9
10	0000	043	086	128	170	212	253	294	334	374	4.1	8.3	12.4	16.6	20.7	24.8	29.0	33.1	37.3
11	414	453	492	531	569	607	645	682	719	755	3.8	7.6	11.3	15.1	18.9	22.7	26.5	30.2	34.0
12	792	828	864	899	934	969	$_0$04	$_0$38	$_0$72	$_1$06	3.5	7.0	10.4	13.9	17.4	20.9	24.3	27.8	31.3
13	1139	173	206	239	271	303	335	367	399	430	3.2	6.4	9.7	12.9	16.1	19.3	22.5	25.7	29.0
14	461	492	523	553	584	614	644	673	703	732	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0
15	1761	790	818	847	875	903	931	959	987	$_0$14	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
16	2041	068	095	122	148	175	201	227	253	279	2.6	5.3	7.9	10.5	13.2	15.8	18.4	21.1	23.7
17	304	330	355	380	405	430	455	480	504	529	2.5	5.0	7.4	9.9	12.4	14.9	17.4	19.9	22.3
18	553	577	601	625	648	672	695	718	742	765	2.3	4.7	7.0	9.4	11.7	14.1	16.4	18.8	21.1
19	788	810	833	856	878	900	923	945	967	989	2.2	4.5	6.7	8.9	11.1	13.4	15.6	17.8	20.0
20	3010	032	054	075	096	118	139	160	181	201	2.1	4.2	6.4	8.5	10.6	12.7	14.8	17.0	19.1
21	222	243	263	284	304	324	345	365	385	404	2.0	4.0	6.1	8.1	10.1	12.1	14.1	16.2	18.2
22	424	444	464	483	502	522	541	560	579	598	1.9	3.9	5.8	7.7	9.7	11.6	13.5	15.4	17.4
23	617	636	655	674	692	711	729	747	766	784	1.8	3.7	5.5	7.4	9.2	11.1	12.9	14.8	16.6
24	802	820	838	856	874	892	909	927	945	962	1.8	3.5	5.3	7.1	8.9	10.6	12.4	14.2	16.0
25	3979	997	$_0$14	$_0$31	$_0$48	$_0$65	$_0$82	$_0$99	$_1$16	$_1$33	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
26	4150	166	183	200	216	232	249	265	281	298	1.6	3.3	4.9	6.6	8.2	9.8	11.5	13.1	14.8
27	314	330	346	362	378	393	409	425	440	456	1.6	3.2	4.7	6.3	7.9	9.5	11.1	12.6	14.2
28	472	487	502	518	533	548	564	579	594	609	1.5	3.0	4.6	6.1	7.6	9.1	10.7	12.2	13.7
29	624	639	654	669	683	698	713	728	742	757	1.5	2.9	4.4	5.9	7.4	8.8	10.3	11.8	13.3
30	4771	786	800	814	829	843	857	871	886	900	1.4	2.8	4.3	5.7	7.1	8.5	10.0	11.4	12.8
31	914	928	942	955	969	983	997	$_0$11	$_0$24	$_0$38	1.4	2.8	4.1	5.5	6.9	8.3	9.7	11.0	12.4
32	5051	065	079	092	105	119	132	145	159	172	1.3	2.7	4.0	5.3	6.7	8.0	9.4	10.7	12.0
33	185	198	211	224	237	250	263	276	289	302	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
34	315	328	340	353	366	378	391	403	416	428	1.3	2.5	3.8	5.0	6.3	7.6	8.8	10.1	11.3
35	5441	453	465	478	490	502	514	527	539	551	1.2	2.4	3.7	4.9	6.1	7.3	8.6	9.8	11.0
36	563	575	587	599	611	623	635	647	658	670	1.2	2.4	3.6	4.8	5.9	7.1	8.3	9.5	10.7
37	682	694	705	717	729	740	752	763	775	786	1.2	2.3	3.5	4.6	5.8	6.9	8.1	9.3	10.4
38	798	309	821	832	843	855	866	877	888	899	1.1	2.3	3.4	4.5	5.6	6.8	7.9	9.0	10.2
39	911	922	933	944	955	966	977	988	999	$_0$10	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
40	6021	031	042	053	064	075	085	096	107	117	1.1	2.1	3.2	4.3	5.4	6.4	7.5	8.6	9.7
41	128	138	149	160	170	180	191	201	212	222	1.0	2.1	3.1	4.2	5.2	6.3	7.3	8.4	9.4
42	232	213	253	263	274	284	294	304	314	325	1.0	2.0	3.1	4.1	5.1	6.1	7.2	8.2	9.2
43	335	345	355	365	375	385	395	405	415	425	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
44	435	444	454	464	474	484	493	503	513	522	1.0	2.0	2.9	3.9	4.9	5.9	6.8	7.8	8.8
45	6532	542	551	561	571	580	590	599	609	618	1.0	1.9	2.9	3.8	4.8	5.7	6.7	7.6	8.6
46	628	637	646	656	665	675	684	693	702	712	0.9	1.9	2.8	3.7	4.7	5.6	6.5	7.5	8.4
47	721	730	739	749	758	767	776	785	794	803	0.9	1.8	2.7	3.7	4.6	5.5	6.4	7.3	8.2
48	812	821	830	839	848	857	866	875	884	893	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
49	902	911	920	928	937	946	955	964	972	981	0.9	1.8	2.6	3.5	4.4	5.3	6.1	7.0	7.9
50	6990	998	$_0$07	$_0$16	$_0$24	$_0$33	$_0$42	$_0$50	$_0$59	$_0$67	0.9	1.7	2.6	3.4	4.3	5.2	6.0	6.9	7.7
51	7076	084	093	101	110	118	126	135	143	162	0.8	1.7	2.5	3.4	4.2	5.1	5.9	6.7	7.6
52	160	168	177	185	193	202	210	218	226	235	0.8	1.7	2.5	3.3	4.1	5.0	5.8	6.6	7.4
53	243	251	259	267	275	284	292	300	308	316	0.8	1.6	2.4	3.2	4.1	4.9	5.7	6.5	7.3
54	324	332	340	348	356	364	372	380	388	396	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2

N	0	1	2	3	4	5	6	7	8	9	PROPORTIONAL PARTS.								
											1	2	3	4	5	6	7	8	9
55	7404	412	419	427	435	443	451	459	466	474	0.8	1.6	2.3	3.1	3.9	4.7	5.5	6.3	7.0
56	482	490	497	505	513	520	528	536	543	551	0.8	1.5	2.3	3.1	3.8	4.6	5.4	6.1	6.9
57	559	566	574	582	589	597	604	612	619	627	0.8	1.5	2.3	3.0	3.8	4.5	5.3	6.0	6.8
58	634	642	649	657	664	672	679	686	694	701	0.7	1.5	2.2	3.0	3.7	4.5	5.2	5.9	6.7
59	709	716	723	731	738	745	752	760	767	774	0.7	1.5	2.2	2.9	3.6	4.4	5.1	5.8	6.6
60	7782	789	796	803	810	818	825	832	839	846	0.7	1.4	2.2	2.9	3.6	4.3	5.0	5.7	6.5
61	853	860	868	875	882	889	896	903	910	917	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.4
62	924	931	938	945	952	959	966	973	980	987	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
63	993	$_0$00	$_0$07	$_0$14	$_0$21	$_0$28	$_0$35	$_0$41	$_0$48	$_0$55	0.7	1.4	2.1	2.7	3.4	4.1	4.8	5.5	6.2
64	8062	069	075	082	089	096	102	109	116	122	0.7	1.3	2.0	2.7	3.4	4.0	4.7	5.4	6.1
65	8129	136	142	149	156	162	169	176	182	189	0.7	1.3	2.0	2.7	3.3	4.0	4.6	5.3	6.0
66	195	202	209	215	222	228	235	241	248	254	0.7	1.3	2.0	2.6	3.3	3.9	4.6	5.2	5.9
67	261	267	274	280	287	293	299	306	312	319	0.6	1.3	1.9	2.6	3.2	3.9	4.5	5.1	5.8
68	325	331	338	344	351	357	363	370	376	382	0.6	1.3	1.9	2.5	3.2	3.8	4.4	5.1	5.7
69	388	395	401	407	414	420	426	432	439	445	0.6	1.2	1.9	2.5	3.1	3.7	4.4	5.0	5.6
70	8451	457	463	470	476	482	488	494	500	506	0.6	1.2	1.8	2.5	3.1	3.7	4.3	4.9	5.5
71	513	519	525	531	537	543	549	555	561	567	0.6	1.2	1.8	2.4	3.0	3.6	4.3	4.9	5.5
72	573	579	585	591	597	603	609	615	621	627	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4
73	633	639	645	651	657	663	669	675	681	686	0.6	1.2	1.8	2.4	3.0	3.5	4.1	4.7	5.3
74	692	698	704	710	716	722	727	733	739	745	0.6	1.2	1.7	2.3	2.9	3.5	4.1	4.7	5.2
75	8751	756	762	768	774	779	785	791	797	802	0.6	1.2	1.7	2.3	2.9	3.5	4.0	4.6	5.2
76	808	814	820	825	831	837	842	848	854	859	0.6	1.1	1.7	2.3	2.8	3.4	4.0	4.5	5.1
77	865	871	876	882	887	893	899	904	910	915	0.6	1.1	1.7	2.2	2.8	3.4	3.9	4.5	5.0
78	921	927	932	938	943	949	954	960	965	971	0.6	1.1	1.7	2.2	2.8	3.3	3.9	4.4	5.0
79	976	982	987	993	998	$_0$04	$_0$09	$_0$15	$_0$20	$_0$25	0.5	1.1	1.6	2.2	2.7	3.3	3.8	4.4	4.9
80	9031	036	042	047	053	058	063	069	074	079	0.5	1.1	1.6	2.2	2.7	3.2	3.8	4.3	4.9
81	085	090	096	101	106	112	117	122	128	133	0.5	1.1	1.6	2.1	2.7	3.2	3.7	4.3	4.8
82	138	143	149	154	159	165	170	175	180	186	0.5	1.1	1.6	2.1	2.6	3.2	3.7	4.2	4.7
83	191	196	201	206	212	217	222	227	232	238	0.5	1.0	1.6	2.1	2.6	3.1	3.6	4.2	4.7
84	243	248	253	258	263	269	274	279	284	289	0.5	1.0	1.5	2.1	2.6	3.1	3.6	4.1	4.6
85	9294	299	304	309	315	320	325	330	335	340	0.5	1.0	1.5	2.0	2.5	3.0	3.6	4.1	4.6
86	345	350	355	360	365	370	375	380	385	390	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
87	395	400	405	410	415	420	425	430	435	440	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
88	445	450	455	460	465	469	474	479	484	489	0.5	1.0	1.5	2.0	2.5	2.9	3.4	3.9	4.4
89	494	499	504	509	513	518	523	528	533	538	0.5	1.0	1.5	1.9	2.4	2.9	3.4	3.9	4.4
90	9542	547	552	557	562	566	571	576	581	586	0.5	1.0	1.4	1.9	2.4	2.9	3.4	3.8	4.3
91	590	595	600	605	609	614	619	624	628	633	0.5	0.9	1.4	1.9	2.4	2.8	3.3	3.8	4.3
92	638	643	647	652	657	661	666	671	675	680	0.5	0.9	1.4	1.9	2.3	2.8	3.3	3.8	4.2
93	685	689	694	699	703	708	713	717	722	727	0.5	0.9	1.4	1.9	2.3	2.8	3.3	3.7	4.2
94	731	736	741	745	750	754	759	763	768	773	0.5	0.9	1.4	1.8	2.3	2.8	3.2	3.7	4.1
95	9777	782	786	791	795	800	805	809	814	818	0.5	0.9	1.4	1.8	2.3	2.7	3.2	3.6	4.1
96	823	827	832	836	841	845	850	854	859	863	0.5	0.9	1.4	1.8	2.3	2.7	3.2	3.6	4.1
97	868	872	877	881	886	890	894	899	903	908	0.4	0.9	1.3	1.8	2.2	2.7	3.1	3.6	4.0
98	912	917	921	926	930	934	939	943	948	952	0.4	0.9	1.3	1.8	2.2	2.6	3.1	3.5	4.0
99	956	961	965	969	974	978	983	987	991	996	0.4	0.9	1.3	1.7	2.2	2.6	3.1	3.5	3.9

$\sqrt{10}$

$\sqrt{20}$

$\sqrt{\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{1}{5}\sqrt{10}$

$\sqrt{\frac{5}{32}} = \sqrt{\frac{10}{64}} = \frac{1}{8}\sqrt{10}$

$\frac{3}{40}\sqrt{10}$